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THE FINITE-VOLUME PARTICLE METHOD ON A MOVING DOMAIN

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Abstract. In the present work we apply the Finite-Volume particle method (FVPM) to a test problem posed on a moving geometry. The FVPM is a relatively new meshless method for discretizing conservation laws, which combines the generic features of a Finite-Volume scheme and a particle method. After a brief derivation of the method, a special Ansatz for the movement of the particles is proposed. Finally we present numerical results obtained for the test problem using FVPM.

1. Introduction

The Finite-Volume Particle Method (FVPM) is a relatively new meshless method for solving hyperbolic systems of conservation laws. The motivation for developing a new scheme was to unify advantages of particles methods and Finite-Volume methods (FVM) in one scheme. The FVPM combines the generic features of a Finite-Volume scheme and a particle method, namely the concept of a numerical flux function and the flow description using moving particles. This method was studied in detail in [1 - 9].

Here we shortly present the application of the FVPM to a test problem posed on a moving domain, a problem which was discussed in detail in [8].

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2. Derivation of the method

We consider conservation laws written in the form

$$\partial_t \boldsymbol{u} + \nabla \cdot \boldsymbol{\mathcal{F}}(\boldsymbol{u}) = \boldsymbol{0}, \quad \forall \boldsymbol{x} \in \Omega(t) \subset \mathbb{R}^2,\, t > 0 \tag{1}$$

with initial conditions $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega(t)$, and suitable boundary conditions, where $\Omega(t)$ is a moving, bounded domain in $\mathbb{R}^2$, $\boldsymbol{u}(\boldsymbol{x}, t) \in \mathbb{R}^m$, $m > 0$ denotes the vector of conservative quantities, and $\boldsymbol{\mathcal{F}}(\boldsymbol{u}(\boldsymbol{x}, t))$ denotes the flux function of the conservation law.

A natural approach to discretize conservation laws is to evaluate the weak formulation of (1) with a discrete set of test functions ψ_i, $i = 1, ..., N$. In classical FVM, the test functions are taken as the characteristic functions $I_{\Omega_i}(x)$ of the control volumes Ω_i on a spatial grid. The discrete quantities are obtained from cell averages. Note that characteristic functions form a partition of unity, i.e. $\sum_{i=1}^{N} I_{\Omega_i}(\boldsymbol{x}) = 1$, $\forall \boldsymbol{x} \in \Omega$.

A similar approach is used in the following, but we introduce a different set of test functions. Since we want to derive a mesh-free method, we should not make use of a mesh. Therefore the conservative variables are approximated at each time step by a finite set of particles located in the spatial domain $\Omega(t)$. From this point of view, the FVPM is a particle method with particle positions $\boldsymbol{x}_i(t)$, which may be irregularly spaced and moving. To each position $\boldsymbol{x}_i(t)$ we associate a function $\psi_i(\boldsymbol{x}, t)$ - the particle. As in the Finite-Volume approach, let $\{\psi_i : i = 1, ..., N\}$ be a partition of unity, but the supports of the functions should overlap. More exactly, we assume that the particles are smooth functions localized around the particle positions $\boldsymbol{x}_i(t)$ and satisfy

$$\sum_{i=1}^{N} \psi_i(\boldsymbol{x}, t) = 1,\, \forall \boldsymbol{x} \in \Omega(t),\, t \geq 0. \tag{2}$$

We construct this partition of unity in the following way:

Taking a Lipschitz continuous function $W : \mathbb{R} \to \mathbb{R}_+$ with compact support (otherwise one has to consider long-range interactions between particles), we define

$$\psi_i(\boldsymbol{x}, t) = \frac{W_i(\boldsymbol{x}, t)}{\sigma(\boldsymbol{x}, t)}, \tag{3}$$

where $\sigma(\boldsymbol{x},t)=\sum_{i=1}^{N}W_i(\boldsymbol{x},t)$, $W_i(\boldsymbol{x},t)=W(\boldsymbol{x}-\boldsymbol{x}_i(t))$, $i=1,...,N$. Such a partition of unity used in FVPM is shown in Figure 1.

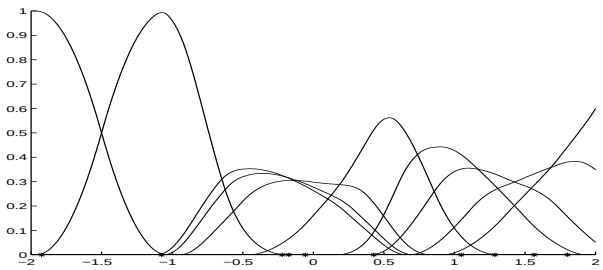

FIGURE 1. A partition of unity used in FVPM

In the FVPM, the particles generically move through the domain, following a prescribed velocity field $\boldsymbol{a}(\boldsymbol{x},t)\in C^0(C^1(\mathbb{R}^2),\mathbb{R}_+)$, i.e. we have $\dot{\boldsymbol{x}}_i=\boldsymbol{a}(\boldsymbol{x}_i,t)$. For $\boldsymbol{a}=\boldsymbol{0}$, one obtains fixed particles, and for $\boldsymbol{a}$ being, for example, the fluid velocity in the case of Euler's equations, one obtains a Lagrangian scheme.

To each particle, one associates a volume $V_i(t)$ and a discrete quantity $\boldsymbol{u}_i(t)$ which is the integral mean value with respect to the test function

$$\boldsymbol{u}_i(t)=\frac{1}{V_i(t)}\int_\Omega \boldsymbol{u}(\boldsymbol{x},t)\psi_i(\boldsymbol{x},t)d\boldsymbol{x},\quad \text{where}\quad V_i(t)=\int_\Omega \psi_i(\boldsymbol{x},t)d\boldsymbol{x}. \tag{4}$$

Testing the conservation law (1) against the new set of test functions $\psi_i(\boldsymbol{x},t)$ and using the quantities defined above, one ends up with a system of ordinary differential equations (see [8] for details)

$$\frac{d}{dt}(V_i\boldsymbol{u}_i)=-\sum_{j=1}^{N}|\boldsymbol{\beta}_{ij}|\boldsymbol{g}_{ij}-\int_{\partial\Omega}\psi_i(\boldsymbol{\mathcal{F}}(\boldsymbol{u})-\boldsymbol{u}\cdot\boldsymbol{b})\cdot\boldsymbol{n}\,d\sigma, \tag{5}$$

with the initial condition

$$\boldsymbol{u}_i(0)=\frac{1}{V_i(0)}\int_\Omega \boldsymbol{u}_0(\boldsymbol{x})\psi_i(\boldsymbol{x},0)d\boldsymbol{x}. \tag{6}$$

Note that the boundary term appears only for particles i which are near the boundary, i.e. $supp\,\psi_i\cap\partial\Omega\neq 0$, and consists of a term containing the flux of the given conservation law, as well of a contribution due to the moving boundary with the velocity $\boldsymbol{b}$.

The coefficients $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{g}_{ij}$ are defined as

$$\boldsymbol{\beta}_{ij}(t) = \boldsymbol{\gamma}_{ij}(t) - \boldsymbol{\gamma}_{ji}(t), \quad \boldsymbol{\gamma}_{ij}(t) = \int_{\Omega(t)} \frac{\psi_i}{\sigma} \nabla W_j d\boldsymbol{x} \tag{7}$$

$$\boldsymbol{g}_{ij}(t) = \boldsymbol{g}(t, \boldsymbol{x}_i, \boldsymbol{u}_i, \boldsymbol{x}_j, \boldsymbol{u}_j, \boldsymbol{n}_{ij}), \quad \boldsymbol{n}_{ij} = \frac{\boldsymbol{\beta}_{ij}}{|\boldsymbol{\beta}_{ij}|} \tag{8}$$

where $\boldsymbol{g}$ is a numerical flux function consistent with the modified flux $\boldsymbol{\mathcal{G}}(t, \dot{\boldsymbol{x}}, \boldsymbol{u})$:

$$\boldsymbol{\mathcal{G}}(t, \dot{\boldsymbol{x}}, \boldsymbol{u}) = \boldsymbol{\mathcal{F}}(\boldsymbol{u}) - \boldsymbol{u} \cdot \dot{\boldsymbol{x}}, \tag{9}$$

where $\dot{\boldsymbol{x}}$ is the particle movement given by $\dot{\boldsymbol{x}}(t) = \boldsymbol{a}(\boldsymbol{x}, t)$. The numerical flux function $\boldsymbol{g}$ can be any numerical flux function used in FVM, but it has to be consistent with the modified flux function $\boldsymbol{\mathcal{G}}$, not with $\boldsymbol{\mathcal{F}}$.

Using an explicit Euler discretization of the time derivative, one obtains

$$V_i^{n+1} \boldsymbol{u}_i^{n+1} = V_i^n \boldsymbol{u}_i^n - \Delta t \sum_{j \in N(i)} |\boldsymbol{\beta}_{ij}^n| \boldsymbol{g}_{ij}^n - \boldsymbol{\mathcal{B}}_i, \tag{10}$$

with $\boldsymbol{u}_i^0 = \frac{1}{V_i^0} \int_\Omega \boldsymbol{u}_0(\boldsymbol{x}) \psi_i(\boldsymbol{x}, 0) d\boldsymbol{x}$, where $\boldsymbol{\mathcal{B}}_i$ is a discretization of the boundary term explained in [8].

A natural reconstruction of a function from the discrete values is given by

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}, t) = \sum_{i=1}^{N} \boldsymbol{u}_i^n \psi_i(\boldsymbol{x}, t) I_{[t_n, t_{n+1})}(t), \quad \boldsymbol{x} \in \Omega,\ t \in [0, T]. \tag{11}$$

3. Special Ansatz for the particles movement

We concentrate on simulating a flow around an oscillating circle in a spatial two-dimensional geometry. The computational domain is given by $\Omega(t) = [0,1] \times [0,1] \setminus B_R(t)$, where $B_R(t) = \{(x,y) \in \mathbb{R}^2 : \|x - x_c(t), y - y_c(t)\| \leq R\}$ is the circle of center $(x_c(t), y_c(t))$ and radius R. Let us denote the domain's boundary by $\partial\Omega(t) := \Gamma_0 \cup \Gamma_R(t)$, where Γ_0 is the exterior boundary and $\Gamma_R(t)$ is the boundary of the moving circle.

We consider a simple, rigid movement of the circle, although one may consider another types of movements. In our example the circle is oscillating up and down, for example

with respect to the following equations:

$$\dot{x}_c(t) = 0, \quad x_c(0) = x_c^0 \tag{12}$$

$$\dot{y}_c(t) = A\omega\cos(\omega t), \quad y_c(0) = y_c^0, \tag{13}$$

where A is *the amplitude of the motion* and ω is *the frequency*.

For the fluid-structure interaction problem which is considered here the effects due to viscosity can be neglected. Hence, the fluid is modeled by Euler's equations for compressible inviscid flow.

In formula (5) there are incorporated two movements: $\boldsymbol{a}$, the movement of the particles (through the numerical flux function $\boldsymbol{g}$), and $\boldsymbol{b}$, the movement of the boundary. Now we have to answer the question: being given the velocity field $\boldsymbol{b}$, how should the particles move?

One may observe that it is not suitable to move the particles with the flow velocity if a smoothly varying particle distribution is desired. Therefore we consider that the movement of the particles $\boldsymbol{a}$ is given by the solution of a Laplace equation with corresponding boundary conditions, namely *zero* velocity at the exterior boundaries and velocity of the circle at the interior boundary:

$$\begin{cases} \Delta\boldsymbol{a}(\boldsymbol{x},t) = \mathbf{0}, & \Omega(t) \\ \boldsymbol{a}(\boldsymbol{x},t) = \mathbf{0}, & \Gamma_0(t) \\ \boldsymbol{a}(\boldsymbol{x},t) = (\dot{x}_c(t), \dot{y}_c(t)), & \Gamma_R(t) \end{cases} \tag{14}$$

In this way the particles follow the domain geometry. In this example, since the movement of the boundary is restricted to a rigid body movement of an isolated object, the whole distribution of particles may be moved with the boundary. In this way the particles remain rigid, i.e. there is no relative motion between the particles. The advantage of this rigid movement is clear, we do not have to recompute every time the coefficients β_{ij} for example. However, the rigid movement approach is less general than the one proposed here.

In [8] we also investigated under which conditions on the motion of the circle and the

smoothing length of the particles no 'holes' are developed in the domain. By a 'hole' we understand a space which is not covered by the support of any particle.

4. **Numerical results**

Here we present numerical results concerning the test problem defined in the previous section.

If the circle moves periodically up and down, like specified in (12), (13), there exists a periodic solution, i.e. after a few oscillations up and down the flow becomes periodic, with the same period as the circle's movement. To see this, we compute the difference between the solution every time when the circle attains its initial position, moving upwards, i.e. exactly after a complete period:

$$e_k = \sum_{i \in N} \left| \rho_i^k V_i^k - \rho_i^{k+1} V_i^{k+1} \right|, \quad k = 0, 1, \ldots, k_{max},$$

where $k_{max} = [T/P]$, $P = 2\pi/\omega$ is the period of the movement, T is the final time, t_0 is the time when the circle starts to move, $\rho_i^k = \rho_i(t_0 + kP)$ and $V_i^k = V_i(t_0 + kP)$. For this computation we choose $N = 50 \times 50$ uniform distributed particles, $t_0 = 0$, $\omega = 10\pi$, $A = 0.1$, $P = 2\pi/\omega = 0.2$, and $T = 4.05$. Hence, $k_{max} = 20$. As can be seen in Figure 2, after around 10 complete oscillations, the differences e_k are so small that the flow can be considered to be periodic.

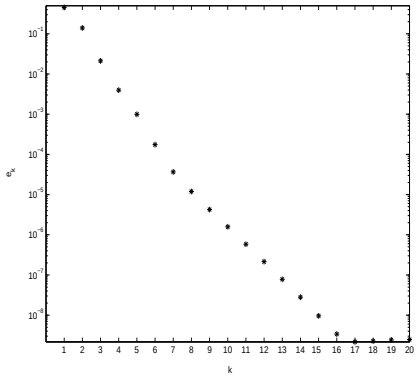

FIGURE 2. Differences e_k versus k

Now we choose $N = 100 \times 100$ quasi-random distributed and moving particles. The movement of the circle is as before, i.e. $A = 0.1$ and $\omega = 10\pi$. The solution at time $T = 0.55$ is presented in Figure 3 and 4. In Figure 3(left) one may see the irregular particle positions together with their corresponding density. The solution reconstructed on a uniform grid is shown in Figure 3(right) (isolines of the density) and Figure 4 (isolines of the velocity components).

These results show that the method works also in the case of a time-dependent domain and using irregular distributed and moving particles.

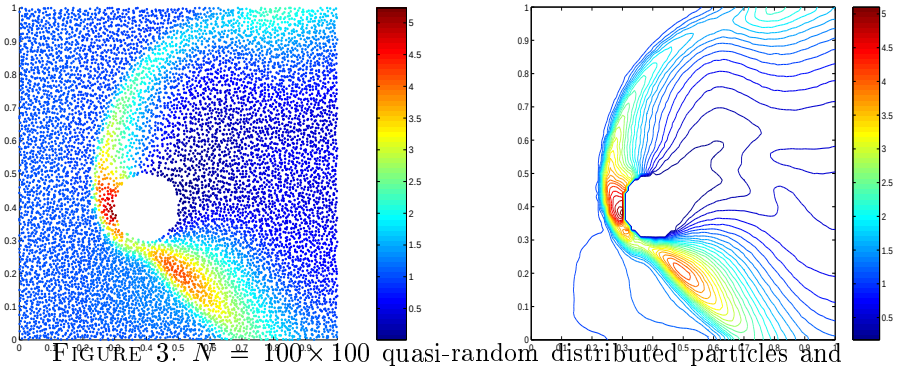

FIGURE 3. $N = 100 \times 100$ quasi-random distributed particles and their corresponding density (left) and isolines of the density (right)

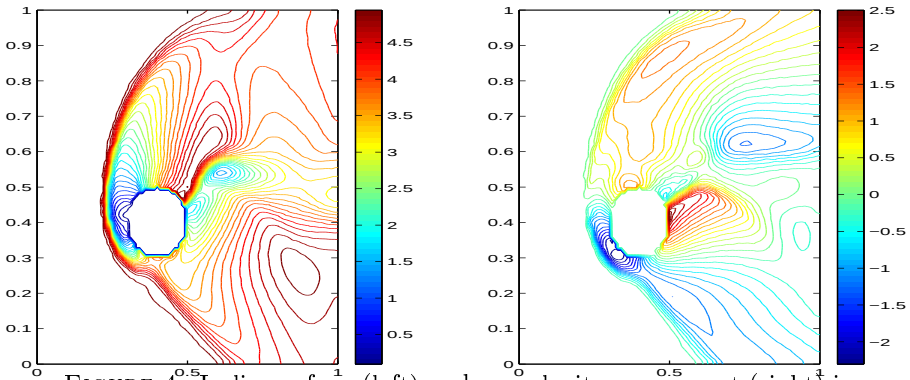

FIGURE 4. Isolines of u - (left) and v - velocity component (right) in the same case as in Figure 3

5. Conclusions

We presented here an application of the FVPM to a spatial two-dimensional problem posed on a moving domain, where the meshless character of the method is fully exploited. The particles are irregularly distributed in the domain and they are moving in a non-Lagrangian way such that they smoothly follow the time-dependent computational domain.

Numerical results indicate that the method is well-suited for such problems. Also the discretization of the boundary conditions works very satisfactory.

Thus, a first step to applying the FVPM to real fluid-structure interaction problems, which in general limit the use of grid-based methods, is done.

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