STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 2, June 2005

POLYNOMIAL REPRODUCIBILITY OF A CLASS OF REFINABLE FUNCTIONS: COEFFICIENTS PROPERTIES

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Abstract. In this paper we investigate some properties of the coefficients of the formulas defining the polynomial reproducibility of a class of refinable functions.

1. **Introduction**

It is well known that constructing refinable approximation operators for real valued functions, it is desiderable to obtain operators approximating smooth functions f with an order of accuracy comparable to the best refinable function approximation.

The key for obtaining operators with such property is to require that they reproduce appropriate classes of polynomials.

Considering the G.P. refinable B-basis on $I = [0, n+1]$, $W_j = \{w_{ji}(x)\}_{i=-n}^{N_j}$, constructed by starting from the class of refinable function defined in [2], it has been proved in [6] that the quasi-interpolatory refinable operators of the form

$$Q_j f(x) = \sum_{i=-n}^{N_j} (\lambda_{ji} f)\, w_{ji}(x) \qquad x \in I = [0, n+1] \tag{1.1}$$

where $N_j = 2^j (n+1) - 1$, and $\{\lambda_{ji}\}_{i=-n}^{N_j}$ is a set of linear functionals, reproduce polynomials $\in \mathbb{P}_l$, the class of polynomials of degree $l-1$, with $1 \le l \le n-1$, if and only if

$$\lambda_{ji} x^{k-1} = \eta_{ji}^{(k)} \quad k = 1, \ldots, l \quad i = -n, \ldots, N_j \tag{1.2}$$

Received by the editors: 12.01.2005.
2000 *Mathematics Subject Classification.* 41A36.
Key words and phrases. Refinable functions, B-splines, optimal bases.

where $\eta_{ji}^{(k)}$ are such that

$$x^{k-1} = \sum_{i=-n}^{N_j} \eta_{ji}^{(k)} w_{ji}(x) \quad x \in I. \tag{1.3}$$

Considering the operators λ_{ji} defined as $\lambda_{ji} := \sum_{k=1}^{l} \alpha_{jik}\lambda_{jik}$ it has been proved in [6], that assuming

$$\alpha_{jik} = \sum_{\nu=0}^{k-1} (-1)^{\nu} \eta_{ji}^{(k-\nu)} symm_{\nu}(\tau_{ji1}, \tau_{ji2}, \dots, \tau_{jik-1}) \quad k = 1, \dots, l \tag{1.4}$$

where $symm_{\nu}(\tau_{ji1}, \tau_{ji2}, \dots, \tau_{jik-1})$ is the symmetric function on the distinct points $\tau_{ji1}, \tau_{ji2}, \dots, \tau_{jik-1} \in I$, the approximating refinable operator:

$$Q_j f(x) = \sum_{i=-n}^{N_j} \sum_{k=1}^{l} \alpha_{jik} [\tau_{ji1}, \tau_{ji2}, \dots, \tau_{jik}] f \, w_{ji}(x) \tag{1.5}$$

where $[\tau_{ji1}, \tau_{ji2}, \dots, \tau_{jik}] f$ is the $k-1$ divided difference of f on the points $\tau_{ji1}, \tau_{ji2}, \dots, \tau_{jik}$ reproduces the polinomials in $\mathbb{P}_l$, $1 \leq l \leq n-1$.

The aim of this paper is to investigate the properties of $\eta_{ji}^{(k)}$ and their localization in the interval I.

We shall make use of such results for proving the convergence of refinable operators (1.1), [7].

The paper is organized as follows. In section 2 we shall give some definitions and the main properties of a wide class of refinable functions Φ_j defined in [2] and we define the B-bases, W_j, constructed starting from Φ_j.

In section 3 we shall consider the evaluation of $\eta_{ji}^{(k)}$ and prove some useful properties.

Section 4 is devoted to prove the relation between the values $\eta_{ji}^{(k)}$ and $\overline{\xi}_{ji}^{(k)}$ that are defined by:

$$x^{k-1} = \sum_{i=-n}^{N_j} \overline{\xi}_{ji}^{(k)} B_{ji}(x), \quad \forall\, x \in I \tag{1.6}$$

where $B_{ji}(x)$ are the normalized B-splines of order $n+1$.

2. Preliminaries

In this section we report for later use some definitions and notations.

It is well known that a scaling function is the solution of a scaling equation

$$\varphi(x) = \sum_{i \in \mathbb{Z}} a_i \varphi(2x - i) \tag{2.1}$$

The class Φ of scaling functions, here considered, consists of the functions $\varphi_h(x)$ solving the scaling equation

$$\varphi_h(x) = \sum_{k=0}^{n+1} a_{kh} \varphi_h(2x - k) \tag{2.2}$$

where h is a real parameter $h \geqslant n \geqslant 2$, and

$$a_{kh} = \frac{1}{2^h}\left[\binom{n+1}{k} + 4\left(2^{h-n} - 1\right)\binom{n-1}{k-1}\right], \quad k = 0, 1, \ldots, n+1. \tag{2.3}$$

Such scaling functions, called G.P. refinable functions, are characterized by the following properties [2]:

(i) supp $\varphi_h = [0, n+1]$;

(ii) $\varphi_h \in C^{n-2}(\mathbb{R})$;

(iii) $\varphi_h(x) > 0 \quad \forall x \in (0, n+1)$;

(iv) φ_h is centrally symmetric, that is $\varphi_h(x) = \varphi_h(n+1-x)$;

(v) its *symbol* $P_n(z) = \sum_{k=0}^{n+1} a_{kh} z^k$ is left-half plane stable;

(vi) $\sum_{k \in \mathbb{Z}} \varphi(x - k) = 1$.

For the above properties, for any admissible h, the system of linearly independent functions

$$\Phi_{0,h} = \{\varphi_h(x - k), \quad k \in \mathbb{Z}\}, \quad h \geq n \geq 2, \tag{2.4}$$

provides a normalized totally positive (NTP) basis in $\mathbb{R}$. Moreover, φ_h generates a multiresolution analysis (MRA) in $L^2(\mathbb{R})$, whose approximating space V_j are defined by

$$V_j = clos_{L^2}\left\{2^{j/2}\varphi_h\left(2^j x - k\right), \ k \in \mathbb{Z}\right\}, \quad j \in \mathbb{Z}^+.$$

Considering a bounded interval $J = [a, b]$, the system

$$\Phi_{j,h} = \left\{ \varphi_{jhk}(x) = 2^{j/2}\varphi_h\left(2^j x - k\right),\ 2^j a - n \leq k \leq 2^j b - 1,\ x \in [a, b] \right\} \tag{2.5}$$

with $j \geq j_0$, where j_0 is the first integer such that $2^{j_0}(b - a) \geq n + 1$, constitutes a NTP basis. For the sake of short notation we shall eliminate the explicit dependence on h of $\Phi_{j,h}$, we write φ_{jk} for φ_{jhk} and we shall set

$$\underline{N}_j = 2^j a - n \quad \overline{N}_j = 2^j b - 1;$$

therefore, Φ_j is a NTP basis for the space $\widetilde{V}_j$ generated on J.

In [3], it has been proved that there exist $\overline{N}_j - \underline{N}_j + 1$ real numbers $\xi_{jk}^{(l)}$ such that,

$$x^{l-1} = \sum_{k=\underline{N}_j}^{\overline{N}_j} \xi_{jk}^{(l)} \varphi_{jk}(x) \quad x \in J, \quad l = 1, \ldots, n-1. \tag{2.6}$$

We remark that for the property (vi) of refinable functions φ_{jk}, there results $\xi_{jk}^{(1)} = 1 \ \forall\ k = \underline{N}_j, \ldots, \overline{N}_j$.

We recall that a TP system of linearly independent functions $W_j = (w_{j0} \ldots w_{jm})$ defined on the bounded interval J is said to be a B-basis (or optimal basis) if each TP basis $U_j = (u_{j0} \ldots u_{jm})$ of the space generated by W_j satisfies the relation

$$U_j = W_j A_j \tag{2.7}$$

where A_j is a non singular TP and stochastic matrix.

In [1] an algorithm for the construction of W_j starting from any U_j is given. This algorithm can be applied, in particular, when the TP system U_j under consideration is constituted by suitable integer shifts of TP refinable functions as considered in (2.5).

Therefore, let $W_j = \{w_{ji}(x)\}_{i=\underline{N}_j}^{\overline{N}_j}$ be the B-basis associated to Φ_j, W_j generates a MRA on J [3]. Moreover, this MRA reproduces polynomials up to the order $d = n - 1$, that is, there exist $\overline{N}_j - \underline{N}_j + 1$ real numbers $\eta_{jk}^{(l)}$ such that

$$x^{l-1} = \sum_{k=\underline{N}_j}^{\overline{N}_j} \eta_{jk}^{(l)} w_{jk}(x) \quad x \in J, \quad l = 1, \ldots, n-1, \tag{2.8}$$

with

$$\underline{\eta}_j^{(l)} = A_j \underline{\xi}_j^{(l)}, \tag{2.9}$$

and A_j is the matrix in (2.7).

By exploiting some properties of the system of functions W_j, we shall establish some properties of the values $\eta_{jk}^{(l)}$ that can be very profitable in proving convergence properties of approximating (in particular quasi-interpolatory) refinable operators.

We assume $J = I = [0, n+1]$ an assumption by no means restrictive, and in such a case there results $\underline{N}_j = -n$, $\overline{N}_j = 2^j (n+1) - 1 = N_j$ and

$$\text{supp } w_{jk} = \left[\max\left(0, \frac{k}{2^j}\right), \min\left(\frac{k+n+1}{2^j}, n+1\right)\right] \tag{2.10}$$

In Fig.1 we show an example of B-basis constructed on $I = [0,6]$ starting from Φ_0, NTP basis with support $[0,6]$ and $h = 10$.

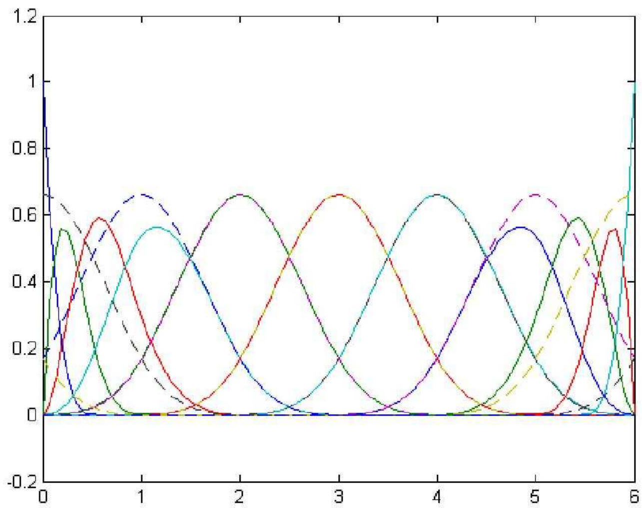

FIGURE 1. The normalized B-bases (solid line) in the interval $[0,6]$. The dashed line represents the starting NTP basis Φ_0 with $n = 5$ and $h = 10$

TABLE 1

	$L=1$	$L=2$	$L=3$
x_i	err	err	err
0.000	5.4 (-17)	7.1 (-16)	5.4 (-15)
1.000	1.1 (-16)	4.4 (-16)	3.7 (-15)
2.000	1.1 (-16)	0.0 (00)	3.3 (-16)
3.000	0.0 (00)	0.0 (00)	1.3 (-16)
4.000	0.0 (00)	1.1 (-16)	2.2 (-16)
5.000	1.8 (-16)	1.4 (-16)	1.1 (-16)
6.000	1.5 (-16)	2.0 (-16)	0.0 (00)

The table 1 shows the relative error, valued in knots x_i, given by the difference between first and second member of (2.8), for several values of L, confirming the polynomial reproducibility.

3. Evaluation and localization of $\eta_{jk}^{(l)}$

In [3] has been proved that for the values $\xi_{0k}^{(l)}$, at level $j=0$, the following relations

$$\xi_{0k}^{(l)} = \sum_{r=0}^{l-1} \binom{l-1}{r} k^{l-1-r} C_r \tag{3.1}$$

$$C_0 = 1, \quad C_1 = \mu_1, \quad C_r = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} \mu_s C_{r-s} \tag{3.2}$$

hold, where $\mu_i := \int_{I\!R} x^i \varphi_0(x)\, dx$ denotes the i-th moment of $\varphi_0(x) = \varphi(x)$.

By rearranging (3.1), we estabilish a formula that allows the evaluation in I of $\xi_{jk}^{(l)}$ for any $j \geqslant 0$, only using the values of $\xi_{jk}^{(2)} = 2^{-j}\left(k + \frac{n+1}{2}\right)$, $k = -n, \ldots, N_j$ and the coefficients C_i $i = 0, \ldots, l-1$ in (3.2). This procedure permits to avoid, for each j, the evaluation of unnecessary values of $\xi_{jk}^{(l)}$, that should be used for the level $j+1$.

Proposition 1. *For $j \geq 0$, there results*

$$\xi_{jk}^{(l)} = \left(\xi_{jk}^{(2)}\right)^{l-1} + 2^{-j(l-1)} \sum_{s=2}^{l-1} \binom{l-1}{s} k^{l-1-s} \left(C_s - C_1^s\right). \tag{3.3}$$

Proof. From (3.1), it is straightforward to verify that for $j=0$

$$\xi_{0k}^{(l)}=k\xi_{0k}^{(l-1)}+k\sum_{r=1}^{l-2}\binom{l-2}{r-1}k^{l-2-r}C_r+C_{l-1},\quad k=-n,\ldots,N_0 \tag{3.4}$$

and in particular, being $\xi_{0k}^{(2)}=k+C_1$, there results

$$\begin{aligned}\xi_{0k}^{(3)} &= k\xi_{0k}^{(2)}+kC_1+C_2=\left(k^2+2kC_1+C_1^2\right)+C_2-C_1^2 \\ &= \left(\xi_{0k}^{(2)}\right)^2+\left(C_2-C_1^2\right),\quad k=-n,\ldots,N_0.\end{aligned} \tag{3.5}$$

By applying iteratively (3.4) and rearranging the terms, we obtain

$$\xi_{0k}^{(l)}=\left(\xi_{0k}^{(2)}\right)^{l-1}+\sum_{s=2}^{l-1}\binom{l-1}{s}k^{l-1-s}\left(C_s-C_1^s\right),\quad k=-n,\ldots,N_0. \tag{3.6}$$

Now considering that

$$\mu_{ji}=2^j\int_{I\!R}\left(2^jx\right)^i\varphi\left(2^jx\right)dx=\int_{I\!R}t^i\varphi\left(t\right)dt=\mu_{0i} \tag{3.7}$$

and

$$\sum_{k\in Z\!\!Z}\left(2^jx-k\right)^i\varphi_{jk}\left(x\right)=\mu_{ji}, \tag{3.8}$$

we obtain, for $l=1,2,\ldots,n-1$

$$x^{l-1}=\sum_{k\in Z\!\!Z}2^{-j(l-1)}\xi_{0k}^{(l)}\varphi_{jk}\left(x\right) \tag{3.9}$$

and we get (3.3) by assuming $\xi_{jk}^{(l)}=2^{-j(l-1)}\xi_{0k}^{(l)}$. □

Therefore the procedure for evaluating for any fixed integer $l,\ 1\le l\le n-1$, $\xi_{jk}^{(l)}$, $k=-n,\ldots,N_j$, consists in the following steps:

- evaluation of $C_i,\ i=0,1,\ldots,l-1$ using (3.2);
- evaluation of $\xi_{jk}^{(2)}=2^{-j}\left(k+\frac{n+1}{2}\right)\quad k=-n,\ldots,N_j$;
- evaluation of $\xi_{jk}^{(l)}$ by means of (3.3).

Once determined $\underline{\xi}_j^{(l)}=\left[\xi_{j,-n}^{(l)}\ldots\xi_{j,N_j}^{(l)}\right]$ we determine by means of (2.9) the vector $\underline{\eta}_j^{(l)}=A_j\underline{\xi}_j^{(l)}$.

We are interested now, in investigate the properties of vector $\underline{\eta}_j^{(l)}$ and the localization of its components, useful for proving the convergence of operators (1.5) as $j\to\infty$.

Starting again from (3.1) and considering that the values of $\xi_{0,k}^{(l)}$, $k = -n, \ldots, N_0$ are obtained evaluating a polynomial of $l-1$ degree with coefficients $C_0, C_1, \ldots, C_{l-1}$ at the point k, and the first derivative of such polynomial evaluated at k coincides with $(l-1)\ \xi_{0,k}^{(l-1)}$ we can write

$$D\underline{\xi}_0^{(l)} = (l-1)\,\underline{\xi}_0^{(l-1)}. \tag{3.10}$$

Therefore we can prove the following

Proposition 2. *All the nodes $\eta_{0,k}^{(l)}$, $k = -n, \ldots, n$, $l = 1, \ldots, n-1$ are non decreasing and they satisfy, for $l > 1$:*

$$0 = \eta_{0,-n}^{(l)} \leq \eta_{0,-n+1}^{(l)} \leq \cdots \leq \eta_{0,-n+l}^{(l)} \leq \cdots \leq \eta_{0,n}^{(l)} = (n+1)^{l-1}. \tag{3.11}$$

Proof. We know that for $l = 1$ there results

$$\eta_{0,k}^{(1)} = 1 \quad k = -n, \ldots, N_0 = n. \tag{3.12}$$

Using (2.8) with $j = 0$, and taking into account that $w_{0k}(0) = \delta_{k,-n}$ and $w_{0k}(n+1) = \delta_{k,n}$ one has, for $1 < l \leq n-1$,

$$0 = \sum_{k=-n}^{n} \eta_{0,k}^{(l)} w_{0k}(0) = \eta_{0,-n}^{(l)}, \quad (n+1)^{l-1} = \sum_{k=-n}^{n} \eta_{0,k}^{(l)} w_{0k}(n+1) = \eta_{0,n}^{(l)}. \tag{3.13}$$

Finally by (2.9) and (3.10)

$$D\underline{\eta}_0^{(l)} = A_0 D\underline{\xi}_0^{(l)} = (l-1)\,A_0\underline{\xi}_0^{(l-1)} = (l-1)\,\underline{\eta}_0^{(l-1)}. \tag{3.14}$$

In [4] has been proved that

$$0 = \eta_{0,-n}^{(2)} \leq \eta_{0,-n+1}^{(2)} \leq \cdots \leq \eta_{0,n}^{(2)} = n+1, \tag{3.15}$$

that can be also deduced considering (3.12) - (3.14).

Therefore, for $l = 3$ the sequence $\eta_{0,i}^{(3)}$ $i = -n, \ldots, n$, using (3.14), is non decreasing and bounded by 0 and $(n+1)^2$. By induction we prove (3.11) for each $1 < l \leq n-1$. □

Remark 1. *We remark that being* $\eta_{0,-n}^{(2)} = 0$*, when* $l = 3$ *one has for (3.13)* $\eta_{0,-n}^{(3)} = 0$ *and using (3.14)* $\eta_{0,-n+1}^{(3)} = 0$*. Then for* $l \geq 2$*, there results*

$$\eta_{0,-n}^{(l)} = \eta_{0,-n+1}^{(l)} = \ldots = \eta_{0,-n+l-2}^{(l)} = 0.$$

The following proposition gives the precise localization of $\eta_{0,i}^{(l)}$ for any $i = -n, \ldots, n$ and $l \geq 2$.

Proposition 3. *There results:*

$$\begin{cases} \eta_{0,i}^{(l)} \in \left[0, (i+n+1)^{l-1}\right] & \text{for } i = -n, \ldots, 0 \\ \\ \eta_{0,i}^{(l)} \in \left[i^{l-1}, (n+1)^{l-1}\right] & \text{for } i = 1, \ldots, n \quad . \end{cases} \tag{3.16}$$

We recall that $[0, i+n+1]$ *are the supports of* w_{0i} *for* $i = -n, \ldots, 0$ *and* $[i, n+1]$ *are that ones of* w_{0i} *for* $i = 1, \ldots, n$.

Proof. Let us demonstrate (3.16) just for $i = -n+1, \ldots, -1$ and for $i = 1, \ldots, n-1$, since for $i = -n, n$ and $i = 0$ it is obvious.

Recalling that W_0 is TP and normalized, for $s = -n+1, \ldots, -1$ using (2.8) and (3.11) with $j = 0$, we obtain

$$x^{l-1} = \sum_{i=-n}^{n} \eta_{0i}^{(l)} w_{0i}(x) \geq \sum_{i=s}^{n} \eta_{0s}^{(l)} w_{0i}(x) = \eta_{0s}^{(l)} \sum_{i=s}^{n} w_{0i}(x).$$

Since $1 = \sum_{i=-n}^{n} w_{0i}(x) = \sum_{i=-n}^{s-1} w_{0i}(x) + \sum_{i=s}^{n} w_{0i}(x)$, one has

$$\sum_{i=s}^{n} w_{0i}(x) = 1 - \sum_{i=-n}^{s-1} w_{0i}(x).$$

If it were $\eta_{0s}^{(l)} > (s+n+1)^{l-1}$, for $x \in [n+s-1, n+s]$ we had

$$x^{l-1} > (s+n+1)^{l-1} (1 - w_{0s-1}(x)) \tag{3.17}$$

since $w_{0k}(x) \equiv 0$, $k = -n, \ldots, s-2$ for $x > n+s-1$.

But $w_{0s-1}(n+s) = 0$, thus in $[n+s-1, n+s]$ there are points x for which $(1 - w_{0s-1}(x)) > \frac{(s+n)^{l-1}}{(s+n+1)^{l-1}}$, and for the same points, (3.17) would give $x^{l-1} > (s+n)^{l-1}$, a contradiction.

Consider now $s = 1, \ldots, n-1$; for each fixed s, supp $w_{0s} = [s, n+1]$. We had to prove that $\eta_{0s}^{(l)} > s^{l-1}$. For $x \in [s, s+1]$, there results

$$x^{l-1} = \sum_{i=-n}^{n} \eta_{0i}^{(l)} w_{0i}(x) = \sum_{i=-n+s}^{s} \eta_{0i}^{(l)} w_{0i}(x) \leq \eta_{0s}^{(l)} \sum_{i=-n+s}^{s} w_{0i}(x) \leq \eta_{0s}^{(l)}.$$

If we were to take $\eta_{0s}^{(l)} < s^{l-1}$, we would have $x^{l-1} < s^{l-1}$ a contradiction. □

For extending the results to the level j we recall relation (2.10) and, by denoting $y_{j,i}$, $y_{j,i+n+1}$ the bounds of supp w_{ji}, we can write:

$$y_{j,i}^{l-1} \leq \eta_{ji}^{(l)} \leq y_{j,i+n+1}^{l-1} \tag{3.18}$$

Remark 2. *In the supports* $[y_{j,i}, y_{j,i+n+1}]$ *there are, for any* i, $n+2$ *uniformly spaced (norm* $\triangle_j = 2^{-j}$ *) points, also partially coinciding with 0 or* $n+1$.

4. Relation between $\eta_{jk}^{(l)}$ and $\xi_{jk}^{(l)}$

In the following Proposition 4 we determine a relation between the values $\eta_{jk}^{(l)}$ and $\overline{\xi}_{jk}^{(l)}$ that are such that:

$$x^{l-1} = \sum_{i=-n}^{N_j} \overline{\xi}_{ji}^{(l)} B_{ji}(x) \quad x \in I, \ 1 \leq l \leq n,$$

where $B_{ji}(x)$ are the normalized B-splines of order $n+1$ and uniformly spaced knots, that, as we know, have the same supports of $w_{ji}(x)$.

Proposition 4. *For any integer* l, $1 \leq l \leq n-1$, *and* $j \geq 0$, *there results*

$$\eta_{ji}^{(l)} = C_j(i,l)\, \overline{\xi}_{ji}^{(l)} \tag{4.1}$$

where $0 \leq C_j(i,l) \leq [2(n+1)]^{l-1}$.

Proof. Let $suppB_{ji} = [y_{j,i}, y_{j,i+n+1}]$ we know [8] that

$$\overline{\xi}_{ji}^{(l)} = \frac{symm_{l-1}(y_{j,i+1}, \ldots, y_{j,i+n})}{\binom{n}{l-1}} \quad i = -n, \ldots, N_j, \tag{4.2}$$

where the symmetric function is defined by:

$$symm_k\left(t_1,t_2,\ldots,t_p\right)=\sum_{1\le i_1\le i_2\le\ldots\le i_k\le p} t_{i_1}t_{i_2}\ldots t_{i_k} \tag{4.3}$$

and the sum is over $\binom{p}{k}$ terms.

For $l=1$ there results: $\overline{\xi}_{ji}^{(1)}=\eta_{ji}^{(1)}=1$, $\forall i=-n,\ldots,N_j$.

Therefore we consider $2\le\ l\ \le\ n-1$ and recall that for each l, $\overline{\xi}_{j,-n}^{(l)}=\overline{\xi}_{j,-n+1}^{(l)}=\ldots=\overline{\xi}_{j,-n+l-2}^{(l)}=0$ and $\eta_{j,-n}^{(l)}=\eta_{j,-n+1}^{(l)}=\ldots=\eta_{j,-n+l-2}^{(l)}=0$, and $\overline{\xi}_{jN_j}^{(l)}=\eta_{jN_j}^{(l)}=(n+1)^{l-1}$.

Then, we can consider the cases: a) $i=-n+l-1,\ldots,-1$; b) $i=0,\ldots,N_j-n$; c) $i=N_j-n+1,\ldots,N_j-1$.

a) Since $y_{j,-n+1}=\ldots=y_{j,0}=0$, $symm_{l-1}\left(y_{j,i+1},\ldots,y_{j,i+n}\right)$ reduces to the sum of $\binom{i+n}{l-1}$ terms, and

$$\binom{i+n}{l-1}\frac{y_{j,1}^{l-1}}{\binom{n}{l-1}}\le\overline{\xi}_{ji}^{(l)}\le\binom{i+n}{l-1}\frac{y_{j,i+n}^{l-1}}{\binom{n}{l-1}}.$$

Therefore

$$0\ =\ \frac{\binom{n}{l-1}}{\binom{i+n}{l-1}}\left(\frac{y_{j,i}}{y_{j,i+n}}\right)^{l-1}\le\frac{\eta_{ji}^{(l)}}{\overline{\xi}_{ji}^{(l)}}\le\left(\frac{y_{j,i+n+1}}{y_{j,1}}\right)^{l-1}\frac{\binom{n}{l-1}}{\binom{i+n}{l-1}}= \tag{4.4}$$

$$\begin{aligned}&=\ (n+1)^{l-1}\frac{n\,(n-1)\ldots(n-l+2)}{(i+n)\,(i+n-1)\ldots(i+n-l+2)}=\\&=\ (n+1)^{l-1}\prod_{h=0}^{l-2}\left(1-\frac{i}{i+n-h}\right)\le\left[2\,(n+1)\right]^{l-1}.\end{aligned}$$

b) In such case

$$\left(\frac{y_{j,i}}{y_{j,i+n}}\right)^{l-1}\le\frac{\eta_{ji}^{(l)}}{\overline{\xi}_{ji}^{(l)}}\le\left(\frac{y_{j,i+n+1}}{y_{j,i+1}}\right)^{l-1}$$

and then

$$0\le\left(1-\frac{n}{i+n}\right)^{l-1}\le\frac{\eta_{ji}^{(l)}}{\overline{\xi}_{ji}^{(l)}}\le(n+1)^{l-1}<\left[2\,(n+1)\right]^{l-1}. \tag{4.5}$$

c) Finally when $i = N_j - n+1, \dots, N_j - 1$, taking into account that $y_{j,N_j+1} = \dots = y_{j,N_j+n+1} = n+1$, there results:

$$\left(\frac{N_j - n + 1}{2^j\,(n+1)}\right)^{l-1} \leq \frac{\eta_{ji}^{(l)}}{\xi_{ji}^{(l)}} \leq \left(\frac{2^j\,(n+1)}{N_j - n + 2}\right)^{l-1};$$

therefore recalling the definition of N_j,

$$0 \leq \frac{\eta_{ji}^{(l)}}{\xi_{ji}^{(l)}} < [2\,(n+1)]^{l-1}. \tag{4.6}$$

By denoting $\frac{\eta_{ji}^{(l)}}{\xi_{ji}^{(l)}} = C_j\,(i,l)$, from (4.4), (4.5), (4.6) we get the thesis. □

Now we show, in Fig. 2, the behaviour of the operator $Q_j f(x)$, given by (1.5), for some values j, for the functions $f(x) = sin(2\pi x)$, and $f(x) = x^4 + |x| * x$. The operator $Q_j f(x)$ has been constructed by starting from $\{\Phi_0\}$ with $n = 5$ and $h = 6$.

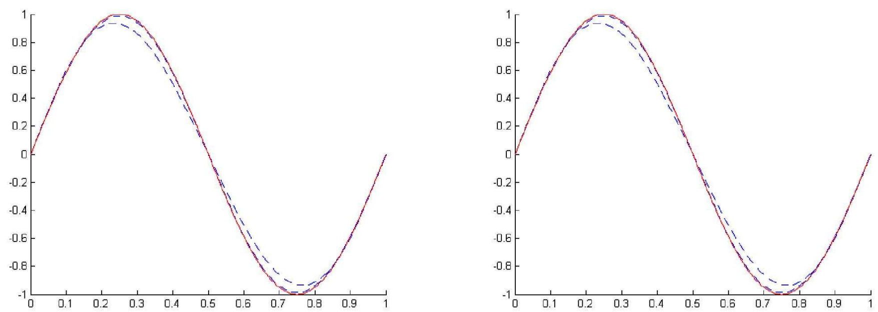

FIGURE 2. The operator $Q_j f(x)$ (dashed line), for some j, for the functions $f(x) = sin(2\pi x)$ (left), and $f(x) = x^4 + |x| * x$ (right).

In the following table 2 we report the infinite norm of the error $|Q_j f - f|$ in $[0,1]$ where $f(x) = sin(2\pi x)$ and $Q_j f$ is constructed by using refinable functions or B-splines of order n+1 in $errQf$ and $errQf\_b$ respectively.

The table 3 is relative to the function $f(x) = x^4 + |x| * x\ x \in [-1,1]$.

TABLE 2

$f(x) = sin(2\pi x)$		
j	$errQf$	$errQf\_b$
3	9.58 (-2)	9.90 (-2)
4	1.25 (-2)	1.34 (-2)
5	8.32 (-4)	8.97 (-4)
6	5.28 (-5)	5.70 (-5)
7	3.32 (-6)	3.58 (-6)
8	2.07 (-7)	2.24 (-7)
9	1.30 (-8)	1.40 (-8)

TABLE 3

$f(x) = x^4 + \|x\| * x$		
j	$errQf$	$errQf\_b$
3	1.17(-2)	1.22(-2)
4	2.31(-3)	2.36(-3)
5	5.37(-4)	5.48(-4)
6	1.32(-4)	1.34(-4)
7	3.28(-5)	3.34(-5)
8	8.20(-6)	8.35(-6)
9	2.05(-6)	2.09(-6)

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