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NOTE ON A TWO-POINT BOUNDARY VALUE PROBLEM UNDER NONRESONANCE CONDITION

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Abstract. The nonresonance method of Mawhin and Ward Jr. is used to discuss the existence of solutions to two point boundary value problems for second order functional-differential equations.

1. Introduction

In this paper we present existence results for the two point boundary value problem

$$\begin{cases} -u''(t) = cu(t) + F(u)(t), & t \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

under the assumption that the constant c is not an eigenvalue of the operator $-u''$ (nonresonance condition) and the growth of $F(u)$ on u is at most linear. More exactly, we will apply the fixed point theorems of Banach, Schauder and the Leray-Schauder principle in order to obtain weak solutions to (1), that is a function $u \in H_0^1(0,1)$ with

$$\int_0^1 u'(t)v'(t)dt = \int_0^1 (cu(t) + F(u)(t))v(t)dt, \quad \text{for all } v \in H_0^1(0,1).$$

The method we use was introduced by J. Mawhin and J. Ward Jr. in [2]. See also [3], [4], [5] for its applications to differential equations. This paper was inspired by [7] and [6], chapter 6. The novelty in this note is that the term $F(u)$ is given by a general operator F from $L^2(0,1)$ to $L^2(0,1)$. In particular, F can be the usual superposition operator $f(t,u(t))$ as in[6] and [7], or a delay operator $f(t,u(t-\tau))$.

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1.1. **Fixed point formulation of problem (1).** We consider $F : L^2(0,1) \to L^2(0,1)$ to be a continuous operator and we define

$$L : H^2(0,1) \cap H_0^1(0,1) \to L^2(0,1), \quad Lu = -u'' - cu$$

Let $L^{-1} : L^2(0,1) \to H^2(0,1) \subset L^2(0,1)$ be the inverse of L. If we look a priori for a solution u of the form $u = L^{-1}v$ with $v \in L^2(0,1)$, then we have to solve the fixed point problem on $L^2(0,1)$:

$$(F \circ L^{-1})(v) = v \tag{2}$$

Throughout this paper we denote:

$$\langle u, v\rangle_{L^2} = \int_0^1 uvdx, \ \ \|u\|_{L^2} = \left(\int_0^1 u^2dx\right)^{1/2}, \ \ \|v\|_{H_0^1} = \left(\int_0^1 (v')^2\, dx\right)^{1/2}$$

1.2. **An auxiliarly result.** We present first an auxiliarly result given in [7]. Let $(\lambda_k)_{k\geq 1}$ be the sequence of all eigenvalues of $-u''$ with respect to the boundary condition $u(0) = u(1) = 0$, and let $(\phi_k)_{k\geq 1}$ be the corresponding eigenfunctions, with $\|\phi_k\|_{L^2} = 1$.

Lemma 1. *Let c be any constant with $c \neq \lambda_k$ for $k = 1, 2, ...$ For each $v \in L^2(0,1)$, there exists a unique weak solution $u \in H_0^1(0,1)$ to the problem*

$$\begin{cases} -u'' - cu = v, \quad \text{on } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

denoted by $L^{-1}v$, and the following eigenfunction expansion holds

$$L^{-1}v = \sum_{k=1}^{\infty} (\lambda_k - c)^{-1} \langle v, \phi_k\rangle_{L^2} \tag{3}$$

where the series converges in $H_0^1(0,1)$. In addition,

$$\left\|L^{-1}v\right\|_{L^2} \leq \mu_c \|v\|_{L^2} \ \text{ for all } v \in L^2(0,1) \tag{4}$$

where

$$\mu_c = \max\left\{|\lambda_k - c|^{-1}; k = 1, 2, ...\right\}.$$

2. **Existence results**

We first show how the fixed point theorems of Banach and Schauder can be used to obtain existence results for problem (1).

Theorem 2. *Suppose*

$$\lambda_j < c < \lambda_{j+1} \text{ for some } j \in \mathbb{N}, j \geq 1, \text{ or } 0 \leq c < \lambda_1 \tag{5}$$

Also assume that

$$\|F(v_1) - F(v_2)\|_{L^2} \leq a \|v_1 - v_2\|_{L^2} \tag{6}$$

for all $v_1, v_2 \in L^2(0,1)$, *where* a *is a nonnegative constant such that*

$$a\mu_c < 1. \tag{7}$$

Then (1) has a unique solution $u \in H_0^1(0,1) \cap H^2(0,1)$. *In addition*

$$(F \circ L^{-1})^n(v_0) \to v \text{ in } L^2(0,1) \text{ as } n \to \infty$$

for any $v_0 \in L^2(0,1)$, *where* $v = Lu$.

Proof. We will show that $F \circ L^{-1}$ is a contraction on $L^2(0,1)$. For this, let $v_1, v_2 \in L^2(0,1)$. Using (6) and (4) we have

$$\left\|F(L^{-1}(v_1)) - F(L^{-1}(v_2))\right\|_{L^2} \leq a \left\|L^{-1}(v_1 - v_2)\right\|_{L^2} \leq a\mu_c \|v_1 - v_2\|_{L^2}.$$

This together with (7) shows that $F \circ L^{-1}$ is a contraction. The conclusion follows from Banach's fixed point theorem.□

Theorem 3. *Suppose that (5) holds,* F *is continuous and satisfies the growth condition*

$$\|F(u)\|_{L^2} \leq a \|u\|_{L^2} + h \tag{8}$$

for all $u \in L^2(0,1)$, *where* $h \in \mathbb{R}_+$ *and* $a \in \mathbb{R}_+$ *is as in (7). Then (1) has at least one solution* $u \in H^2(0,1) \cap H_0^1(0,1)$.

Proof. We have $F \circ L^{-1} = F \circ J \circ L_0^{-1}$ where

$$\begin{cases} L_0^{-1} : L^2(0,1) \to H^2(0,1),\ L_0^{-1}u = L^{-1}u \text{ and} \\ \\ J : H_0^1(0,1) \to L^2(0,1),\ Ju = u. \end{cases}$$

Recall that F is continuous and by (8) is bounded. Next, by Rellich-Kondrachov theorem (see [1]), the imbedding of $H_0^1(0,1)$ into $L^2(0,1)$ is completely continuous. Thus, $F \circ L^{-1}$ is a completely continuous operator. On the other hand, from (8) and (4) we have

$$\left\| F(L^{-1}(v)) \right\|_{L^2} \leq a \left\| L^{-1}(v) \right\|_{L^2} + h \leq a\mu_c \left\| v \right\|_{L^2} + h.$$

Now (7) guarantees that $F \circ L^{-1}$ is a self-map of a sufficiently large closed ball of $L^2(0,1)$. Thus we may apply Schauder's fixed point theorem.□

Better results can be obtained if we use the Leray-Schauder principle (see [6]).

Theorem 4. *Suppose that F is continuous and has the decomposition*

$$F(u) = G(u)u + F_0(u) + F_1(u)$$

Also assume that

$$\left\| F_0(u) \right\|_{L^2} \leq a \left\| u \right\|_{L^2} + h_0 \tag{9}$$

$$\left\| F_1(u) \right\|_{L^2} \leq b \left\| u \right\|_{L^2} + h_1 \tag{10}$$

$$\langle u, F_1(u) \rangle_{L^2} \leq 0 \tag{11}$$

$$-M \leq G(u)(t) + c \leq \beta < \lambda_1 \tag{12}$$

for all $u \in L^2(0,1)$, where $a, b, h_0, h_1, M, \beta \in \mathbb{R}_+$. In addition assume that $0 \leq c \leq \beta$ and

$$a/\lambda_1 < 1 - \beta/\lambda_1. \tag{13}$$

Then (1) has at least one solution $u \in H^2(0,1) \cap H_0^1(0,1)$.

Proof. We look for a fixed point $v \in L^2(0,1)$ of $F \circ L^{-1}$. As above, $F \circ L^{-1}$ is a completely continuous operator. We will show that the set of all solutions to

$$v = \lambda (F \circ L^{-1})(v) \tag{14}$$

when $\lambda \in [0,1]$ is bounded in $L^2(0,1)$. Let $v \in L^2(0,1)$ be any solution of (14). Let $u = L^{-1}v$. It is clear that u solves

$$\begin{cases} -u''(t) - cu(t) = \lambda F(u)(t), \quad t \in (0,1) \\ u(0) = u(1) = 0. \end{cases} \tag{15}$$

Since u is a weak solution of (15), we have

$$\|u\|^2_{H^1_0} = \langle cu + \lambda F(u), u \rangle_{L^2}.$$

It is easy to check that

$$\langle cu + \lambda G(u)u, u \rangle_{L^2} \leq \beta \|u\|_2^2. \tag{16}$$

We define

$$R(u) := \|u\|^2_{H^1_0} - \beta \|u\|^2_2 \tag{17}$$

and using (11), (16) and $c \leq \beta$, we obtain

$$R(u) \leq \|u\|^2_{H^1_0} - \langle cu + \lambda G(u)u, u \rangle_{L^2} \leq |\langle F_0(u), u \rangle_{L^2}|.$$

On the other hand, if we denote $c_k = \langle u, \phi_k \rangle_{L^2} = \langle u, \phi_k \rangle_{H^1_0} / \lambda_k$, we see that

$$\begin{aligned} R(u) &= \sum_{k=1}^{\infty} (\lambda_k - \beta) c_k^2 \geq \sum_{k=1}^{\infty} \lambda_k (1 - \beta/\lambda_1) c_k^2 \\ &\geq (1 - \beta/\lambda_1) \|u\|^2_{H^1_0}. \end{aligned} \tag{18}$$

Recall that

$$\lambda_1 = \inf \left\{ \|u\|^2_{H^1_0} / \|u\|^2_2 \, ; u \in H^1_0(0,1) \setminus \{0\} \right\}$$

and using (18), (17), (9) and Holder's inequality we obtain

$$\begin{aligned} (1 - \beta/\lambda_1) \|u\|^2_{H^1_0} &\leq |\langle F_0(u), u \rangle_{L^2}| \leq \|F_0(u)\|_{L^2} \|u\|_{L^2} \leq a \|u\|^2_{L^2} + h_0 \|u\|_{L^2} \\ &\leq \frac{a}{\lambda_1} \|u\|^2_{H^1_0} + C \|u\|_{H^1_0} \end{aligned}$$

for some constant $C > 0$. Thus (13) guarantees that there is a constant $r > 0$ independent of λ with $\|u\|_{H_0^1} \leq r$. Finally, a bound for $\|v\|_{L^2}$ can be immediately derived from $u = L^{-1}v$. The colnclusion now follows from the Leray-Schauder principle. □

3. **Particular cases**

Particular case 1. Let $F(u)$ be the usual superposition operator, $F(u)(t) = f(t, u(t))$. Then for the problem

$$\begin{cases} -u''(t) = cu(t) + f(t, u(t)), \ t \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{19}$$

we have the following existence result given in [7]:

Theorem 5. *Assume that $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function, $f(\cdot, 0) \in L^2(0,1)$ and that f satisfies the Lipschitz condition*

$$|f(t, v_1) - f(t, v_2)| \leq a\, |v_1 - v_2| \tag{20}$$

for every $v_1, v_2 \in \mathbb{R}$, $t \in (0,1)$ and some $a \geq 0$. Also assume that the conditions (5) and (7) from Thorem 2 are satisfied.

Then (19) has a unique solution $u \in H_0^1(0,1) \cap H^2(0,1)$.

Proof. Using (20) we deduce

$$|f(t,u)| \leq |f(t,u) - f(t,0)| + |f(t,0)| \leq a\,|u| + |f(t,0)|$$

for every $u \in \mathbb{R}$ and $t \in (0,1)$. Moreover, f beeing a Caratheodory function, we have that the Nemitskii operator

$$u \longmapsto f(\cdot, u(\cdot))$$

is well defined, bounded and continuous from $L^2(0,1)$ into $L^2(0,1)$. Using again (20) we obtain

$$\int_0^1 |f(t, v_1(t)) - f(t, v_2(t))|^2\, dt \leq a^2 \int_0^1 |v_1(t) - v_2(t)|^2\, dt$$

so

$$\|F(v_1) - F(v_2)\|_{L^2} \leq a\, \|v_1 - v_2\|_{L^2}\,.$$

The conclusion follows now by applying Theorem 2. □

Particular case 2. Let $0 < \tau < 1$ and let F be defined by

$$F(u)(t) = \begin{cases} f(t, u(t-\tau)), & \tau < t < 1 \\ g(t), & 0 < t < \tau. \end{cases} \tag{21}$$

Theorem 6. *Assume that* $f : (\tau, 1) \times \mathbb{R} \to \mathbb{R}$ *is a Caratheodory function,* $f(\cdot, 0) \in L^2(\tau, 1)$ *and that* f *satisfies the Lipschitz condition*

$$|f(t, v_1) - f(t, v_2)| \leq a\,|v_1 - v_2| \tag{22}$$

for all $v_1, v_2 \in \mathbb{R}$, $t \in (\tau, 1)$ *and some* $a > 0$. *Also assume that* $g \in L^2(0, \tau)$ *and that the conditions (5) and (7) from Theorem 2 are satisfied.*

Then (1) with F *defined by (21) has a unique solution* $u \in H_0^1(0,1) \cap H^2(0,1)$.

Proof. Let $u \in L^2(0,1)$. Then $u(\cdot - \tau) \in L^2(\tau, 1)$. Hence, $f(\cdot, u(\cdot - \tau)) \in L^2(\tau, 1)$. Moreover, since $g \in L^2(0, \tau)$ we have $F(u) \in L^2(0,1)$ is well defined as operator from $L^2(0,1)$ into $L^2(0,1)$.

Let (u_k) be a sequence wich converges to u in $L^2(0,1)$. Let $v_k(t) = u_k(t-\tau)$ and $v(t) = u(t-\tau)$. Then

$$\begin{aligned} \int_\tau^1 (v_k(t) - v(t))^2 dt &= \int_\tau^1 (u_k(t-\tau) - u(t-\tau))^2 dt \\ &= \int_0^{1-\tau} (u_k(t) - u(t))^2 dt \longrightarrow 0, \text{ as } k \to \infty, \end{aligned}$$

so $v_k \to v$ in $L^2(\tau, 1)$ as $k \to \infty$. Consequently, $f(\cdot, v_k(\cdot)) \longrightarrow f(\cdot, v(\cdot))$ in $L^2(\tau, 1)$ and by the definition of F it follows that $F(u_k) \to F(u)$ in $L^2(0,1)$. Using (22) we deduce

$$\begin{aligned} \int_0^1 (F(v_1)(t) - F(v_2)(t))^2 dt &\leq \int_\tau^1 (f(t, v_1(t-\tau)) - f(t, v_2(t-\tau)))^2 dt \\ &\leq a^2 \int_\tau^1 (v_1(t-\tau) - v_2(t-\tau))^2 dt \\ &\leq a^2 \int_0^{1-\tau} (v_1(s) - v_2(s))^2 ds \\ &\leq a^2 \int_0^1 (v_1(s) - v_2(s))^2 ds \end{aligned}$$

and finally

$$\|F(v_1) - F(v_2)\|_{L^2} \leq a \|v_1 - v_2\|_{L^2}.$$

The conclusion follows now by applying Theorem 2. □

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