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COMPARISON BETWEEN DIFFERENT HARVESTING MODELS FOR NON-LINEAR AGE STRUCTURED FISH POPULATIONS

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Abstract. The role of harvesting in discrete nonlinear age-structured fish population models has been studied. The overcompensatory Ricker recruitment function is considered in our model. We show numerically that the maximum sustainable yield (MSY) in harvesting with nets is different very little from (MSY) in selective harvesting. Our models contain a large number of parameters such as mortality, Von Bertalanffy growth parameter and recruitment parameters. The influence of mortality has been studied. The age structured matrix model (general Leslie model) for description of harvesting population dynamics has been used because most marine fish exhibit a clear yearly cycle of spawning, recruitment, migration and growth.

1. **Introduction**

Our basic model is a nonlinear discrete age-structured population model:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}_{t+1} = \begin{pmatrix} 0 & \alpha f_2 r(P) & \cdots & \alpha f_m r(P) \\ \tau_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \tau_{m-1} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}_t \tag{1}$$

The model is based on general biological principles and contains a large number of parameters and functions. Concrete data and further informations will be used to reduce the number of parameters and functions and determine the range of critical

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parameters. Properties of this model class will be investigated and compared with observations. All models are deterministic. The ultimate goal of these models is to obtain precise informations about the state of the species. These informations can then be used to generate recommendations on catches, quotas and equipments.

The models are of the Leslie type and the only nonlinearity is the recruitment function, which we choose to be of Ricker type. Actual date give little support for the precise form of the recruitment function. Otherwise the parameters and functions used are chosen as to reflect concrete marine fish species. In the classical paper by Levin and Goodyear [2], model (1) was used in order to investigate the dynamics of the striped bass in the Hudson river. Such models have been described in many articles, cf. Cushing [1], Getz and Haight [3] and a classical paper by Leslie [4]. In such a model, time is considered as a discrete variable, measured in years. It is most sensible to identify the beginning of the year with spawning. Since selective harvesting is an unrealistic idealization, we concentrate on net harvesting and study in particular the role of the mesh width of fishing nets.

The goal of this paper is to compare between selective harvesting and harvesting with nets. Also the influence mortality on our models is investigated. The plane of the paper is as follows: In Section 2 we present the selective harvesting model while harvesting with nets is presented in Section 3. In Section 4 we give the results of our investigations by using haddock as a numerical example and finally in Section 5, we state the conclusions.

2. The selective harvesting model

Let $x_i(t)$, be the i-th age class of a fish population at time t. Denote the corresponding fecundity by f_i . Next, we let each age class i be exposed to harvesting with constant harvest rate $h_i, i=$ 2,3,...,m, i.e., there is no harvesting in the first class, where m is the maximum age class. So, the model after harvesting has the matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}_{t+1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1-h_2 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \tau_{m-1} & 1-h_m \end{pmatrix} \cdot \begin{pmatrix} 0 & \alpha f_2 r(P) & \cdots & \alpha f_m r(P) \\ \tau_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \tau_{m-1} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}_t \qquad (2)$$

or in vector form, $\underline{x}(t+1) = (I-H)A\underline{x}(t)$, where $P(t)$ is the number of recruits (new borns) of fish from all age classes at time t i.e.,

$$P(t) = \sum_{i=2}^{m} f_i x_i(t)$$

$\tau_i = 1 - \mu_i$ is the density independent probability of survival from age class i to age class $i+1$, where μ_i is the mortality rate of age class i. Finally $r(P)$ describes the recruitment process. $r(P)$ is is a non-negative monotonically decreasing function and α is the productivity parameter, representing the probability of survival of eggs at low densities.

Under the assumptions above, the components of equilibrium vector of (2) are:

$$x_1^* = \frac{L_1 r^{-1}\left(\dfrac{1}{n(h_j)}\right)}{n(h_j)}, \qquad x_2^* = \frac{L_2 H_2 r^{-1}\left(\dfrac{1}{n(h_j)}\right)}{n(h_j)}$$

and generally,

$$x_i^* = \frac{L_i H_i r^{-1}\left(\frac{1}{n(h_j)}\right)}{n(h_j)}, \qquad i = 1, 2, \ldots, m \qquad (3)$$

where

$L_i = \prod_{j=1}^{i} \tau_{j-1}$ is the survival probability from age class 1 to age class i,
$(L_1 = 1), H_1 = 1, H_i = (1-h_2)(1-h_3)\cdots(1-h_i), \qquad i = 2, 3, \cdots, m$

and

$$n(h_i) = \sum_{i=2}^{m} \alpha f_i L_i H_i$$

is called "*the net reproductive number*" because biologically it gives the expected number of offspring per individual over its life time, cf., Cushing and Yicang [5]. Finally the yield is defined by:

$$Y(h_i) = \sum_{i=2}^{m} \frac{w_i h_i x_i^*}{(1-h_i)},$$

where w_i is the growth weight of adult fish and it is described by *Von Bertalanffy growth equation* [8].

$$w(t) = w_\infty (1 - e^{-K(t-\widetilde{t})^3}$$

where K is the rate at which growth weight tends towards its asymptotic value and $\widetilde{t}$ is the age at which growth weight starts.

So from equation (3), we get that the yield in selective harvesting is:

$$Y(h_i) = \frac{r^{-1}\left(\frac{1}{n(h_i)}\right)}{n(h_i)} \sum_{i=2}^{m} w_i h_i L_i H_{i-1}, \tag{4}$$

The maximum sustainable yield is now

$$\max_h Y(h) = Y_{\max}.$$

Reed [6] showed that for selective harvesting, the optimal policy is of the "two-age" type. This means that if we define $j(t)$ recursively by:

$$j(1) = \arg_j \max \frac{w_j L_j}{\sum_{i=j}^{m} f_i L_i}$$

and

$$j(t+1) = \arg_j \max \frac{w_j L_j - w_{j(t)} L_{j(t)}}{\sum_{i=j}^{j(t)-1} f_i L_i}, \ t = 1, 2, \cdots$$

There is a partial harvest at age $j(t+1)$ and a total harvest at age $j(t)$ where $j(t) > j(t+1)$. If we consider that the fecundity is proportional to the weight and assume the mortality is constant, one gets:

$$j(t) = m - (t-1), t = 1, 2, \cdots .$$

If fecundity is proportional to the weight and the mortality is increasing with age, then $j(1) = m$ and $j(t+1), t = 1, 2, \cdots$ are generally smaller than $j(t+1)$ in constant mortality, c.f. Mostafa K. S. Mohamed [7].

The maximum sustainable yield for selective harvesting is:

$$\begin{aligned} Y(h_{j(t+1)}) &= \frac{r^{-1}\left(\dfrac{1}{n(h_{j(t+1)})}\right)}{n(h_{j(t+1)})} \prod_{j=1}^{j(t+1)} \tau_{j-1} \cdot \\ & \left\{ w_{j(t)}(1-h_{j(t+1)}) \prod_{j=j(t+1)+1}^{j(t)} \tau_{j-1} + w_{j(t+1)} h_{j(t+1)} \right\} \qquad (5) \end{aligned}$$

where $r^{-1}(x)$ is given from the stock-recruitment relationships and $h_{j(t+1)}$ is partial harvesting intensity. It can not determined analytically.

We will use the Ricker recruitment relationship as an example. Thus

$$r(x) = e^{-\beta x}, \qquad r^{-1}\left(\frac{1}{n(h)}\right) = \frac{Ln(n(h))}{\beta}$$

Note that in this case, we use a normalized formula for $r(P)$ i.e., $r(0) = 1$.

3. **Harvesting with nets**

If we want to model fishing with nets, we have to translate the width of fishing nets into this model. This will be done as follows. We write $H = h\ diag(0, 0, \cdots, \gamma, 1, \cdots, 1)$ to describe the situation where all fish from class $k+1$ or more are caught, while all fish of class $k-1, k-2, \cdots$ can escape and fish of class k only a fraction $0 \le \gamma \le 1$ is retained. By this we mean that the mesh width is too small for fish from class $k+1$. With the term γ, we can model the fact that the mesh width is a continuous variable.

Now we can use the formulae from selective harvesting with

$$h_i = \begin{cases} 0, & 1 \le i \le k-1 \\ \gamma h, & i = k \\ h, & k \le i \le m \end{cases} \tag{6}$$

The components of equilibrium vector are:

$$\overline{x_i}(h) = \frac{L_i(h)}{n(h)}.r^{-1}\left(\frac{1}{n(h_i)}\right), \qquad i = 1, 2, \cdots, m$$

where

$$n(h) = \sum_{i=1}^{m} \alpha f_i.L_i(h)$$

and

$$L_i(h) = \begin{cases} \prod_{l=1}^{i} \tau_{l-1} & 1 \le i \le k-1 \\ \prod_{l=1}^{k-1} \tau_{l-1}\tau_{k-1}(1-\gamma h) & i = k \\ \prod_{l=1}^{k-1} \tau_{l-1}\tau_{k-1}(1-\gamma h) \prod_{j=k+1}^{i} \tau_{j-1}(1-h)^{j-k}, & k < i \le m. \end{cases}$$

The yield in net harvesting is then

$$Y(h) = \frac{C\ h\ r^{-1}\left(\frac{1}{n(h)}\right)}{n(h)} \left\{ w_k\ \gamma + \sum_{i=k+1}^{m} w_i\ (1-\gamma\ h) \prod_{j=k+1}^{i} \tau_{j-1}\ (1-h)^{j-k-1} \right\} \tag{7}$$

where

$$C = \prod_{l=1}^{k-1} \tau_l$$

We will consider for simplicity that $\gamma = 0$ only for all numerical computations, so equation (7) will be:

$$Y(h) = \frac{h\ r^{-1}\left(\frac{1}{n(h)}\right)}{n(h)} \left\{ w_{k+1}\ l_{k+1} + \sum_{i=k+2}^{m} w_i\ l_i(1-h)^{i-k-1} \right\}$$

where,

$$n(h) = \sum_{i=1}^{k} \alpha f_i l_i + \sum_{i=k+1}^{m} \alpha f_i l_i (1-h)^{i-k}$$

and

$$l_{i+1} = \prod_{j=0}^{i} \tau_j, \qquad l_1 = \tau_o = 1$$

One can show that the function Y has a unique maximum by deriving:

$$\frac{d^2Y}{dh^2} < 0.$$

4. **A numerical example**

Now, we will study the optimal harvesting for haddock using the Leslie model with $\gamma = 0$. The aim is to find a relation between optimal harvesting and beginning class of harvesting " k " for those fish species, k is a discrete parameter measuring the width of the meshes. Large k corresponding to large width. Also we will compare between selective harvesting and harvesting with nets. We showed in section (2) that partial harvesting $h_{j(t+1)}$ can't be determined analytically, so we will determine it numerically.

From Beverton-Holt [8], the maximum age of haddock is $m = 20$ years and it has a constant natural mortality of about 0.2 per year. In order to study the influence of the mortality on our models, we assume that the mortality is increasing as an example in the form

$$\mu(i) = \begin{cases} 0.2, & i \leq \frac{m}{2} \\ \frac{0.4 \times i}{m}, & i > \frac{m}{2} \end{cases}$$

The weight of haddock is determined from the formula

$$w(t) = 1.34 \times (1 - e^{-0.26(t+0.75)})^3 \qquad \text{kg}$$

and the fecundity which proportional to the weight, is determined from

$$f(t) = w(t) \cdot 10^5$$

Ricker stock-recruitment parameters for haddock are

$$\beta = \frac{1}{61.4}, \qquad \alpha = 1.53 \times 10^{-8}$$

4.1. **Influence of mortality.**

4.1.1. *Selective harvesting.*

The influence of parameters in selective and net harvesting are determined from equations (5)and (7) respectively

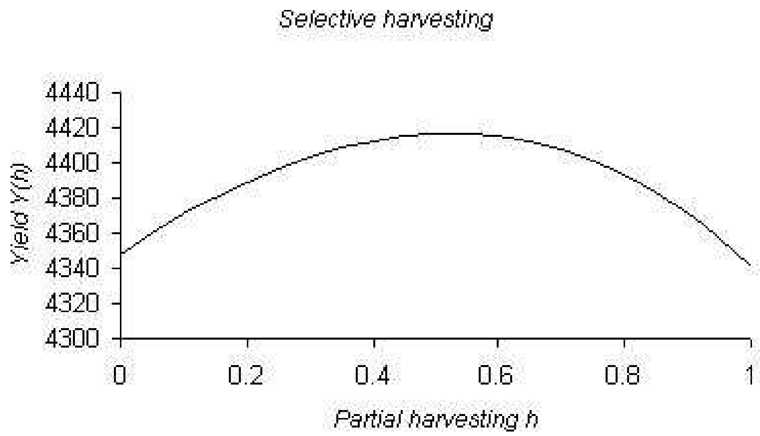


FIGURE 1. Maximum harvesting $h_{j(t+1)} = 0.7$ when constant mortality=0.2, $j(t+1) = 7$ years and MSY= 4407 gm

4.1.2. *Net harvesting.*

mortality: MSY= 4562.81 gm, variable mortality: MSY= 4562.81 gm

4.2. **Influence of Von Bertlanffy parameter K.**

4.2.1. *Selective harvesting.*

4.2.2. *Net harvesting.*

when $K = 0.2$, optimal mesh width $k = 9$ and MSY = 1874 gm,

when $K = 0.3$, optimal mesh width $k = 6$ and MSY = 4447 gm,

when $K = 0.4$, optimal mesh width $k = 5$ and MSY = 6809 gm

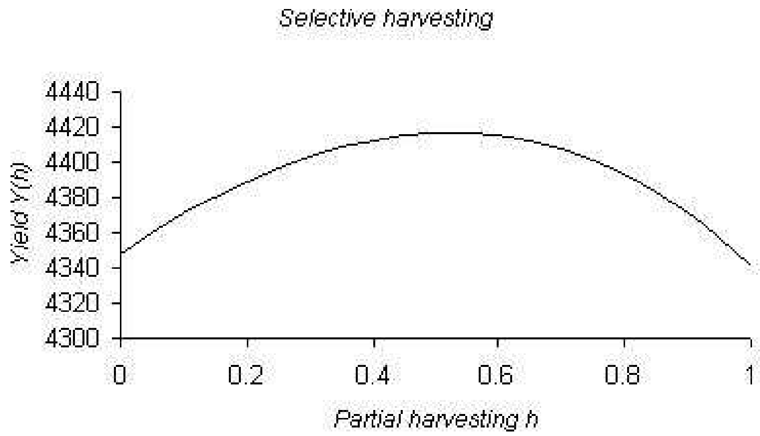


FIGURE 2. Maximum harvesting $h_{j(t+1)} = 0.7$, variable mortality, $j(t+1) = 7$ years and MSY= 4407 gm

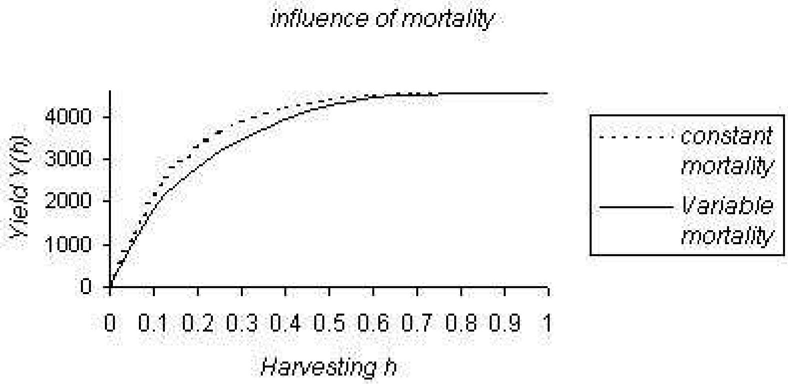
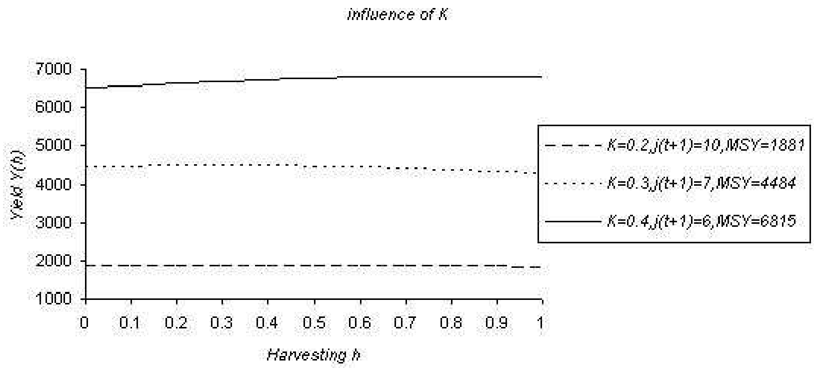


FIGURE 3. Optimal mesh width $k = 6$, $h_{\max} = 0.9$, constant mortality: MSY= 4562.81 gm, variable mortality: MSY= 4562.81 gm

5. Conclusion and results

The purpose of this paper is to compare between two cases of harvesting from a discrete nonlinear age-structured fish population with Ricker stock recruitment function (cf. the classical paper by Ricker [10] and Ricker [9]) and the influence



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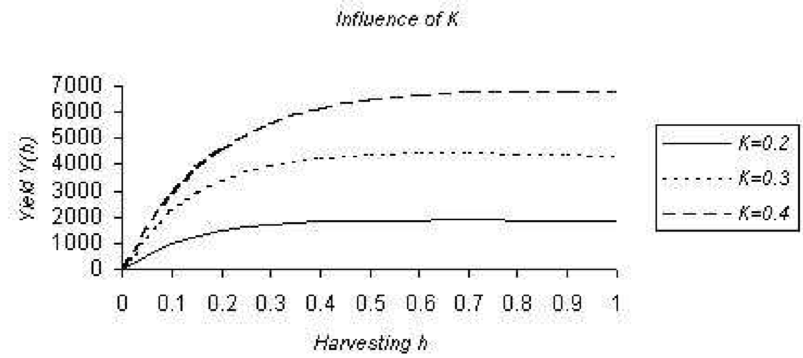

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of paremetrs in our model. In our models, there are many parameters acting on the results such as mortality μ and von-Bertalanffy parameter K. The influence of mortality is that, in general, increasing mortality means the numbers of individual at high age classes are decreasing and MSY in this case is decreasing because the probability of dying is increasing. In our particular example, Figures 1 and 2 indicate that the influence of mortality in selective harvesting is ignored because $j(t+1)$ is less than $m/2$ and mortality is constant in this case. In Figure 3, the influence of mortality in net harvesting is that the values of MSY in variable mortality is slightly smaller than in constant mortality because when mortality is increasing, the survival probability L_i is decreasing i.e., the number of fish which arrive to fishable age is also decreasing. So the influence of mortality parameter on our models is small and we can use a constant mortality as a simplification of models. The influence of the Von-Bertalanffy growth parameter$K$ is that when K increases, the growth function arises to its asymptotic value more quickly, this means that weight is increasing more quickly too and since the heavier fish are more catchable, so the values of $j(t+1)$and optimal mesh width k are decreasing and the value of MSY is increasing as shown in Figures 4 and 5.

The main conclusion is that the MSY in net harvesting is slightly smaller than the MSY in selective harvesting. This is because in selective harvesting, the MSY is over a cube with $m-1$ dimension (the values of $h_i, h_1 = 0$) but in harvesting with nets, the MSY is over a subsets of that cube. These subsets are lines of diagonal of that cube.

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