STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 2, June 2005

MIXED FUNCTIONAL DIFFERENTIAL EQUATION WITH PARAMETER

VERONICA ANA DÂRZU ILEA

Abstract. In this paper we study some special functional differential equation of mixed type. First we take an equation with some initial condition:

$$-\lambda x\prime(t,\lambda) = f(t, x(t,\lambda), x(t-h_1,\lambda), x(t+h_2,\lambda)), t \in R \qquad (1)$$

$$x(t,\lambda) = \varphi(t,\lambda), \quad t \in [t_0 - h_1, t_0 + h_2], \qquad (2)$$

where $h_1,\ h_2 > 0,\ \varphi \in C^1[t_0 - h_1, t_0 + h_2],\ \lambda \in R$ parameter.

The problem that we are interested in is the convergence of the solution of the problem (1)+(2) in the case $\lambda \neq 0$ to the solution of the same problem in case $\lambda = 0$. As an example of this problem we give the linear case of functional differential equation of mixed type. At the end of the article we discuss some inequalities between the solution of equation (1), inequalities that depend on λ.

1. Introduction

In this section we'll discuss the linear case of the mixed functional differential equation (MFDE) with parameter .

Let us consider the problem:

$$-\lambda x\prime(t,\lambda) = \alpha x(t,\lambda) + \beta x(t-h_1,\lambda) + \gamma x(t+h_2,\lambda), \quad t \in R \qquad (3)$$

$$x(t,\lambda) = \varphi(t,\lambda), \quad t \in [t_0 - h_1, t_0 + h_2],\ \varphi \in C^1[t_0 - h_1, t_0 + h_2] \qquad (4)$$

Received by the editors: 26.10.2004.

2000 *Mathematics Subject Classification.* 34K07.

Key words and phrases. Functional differential equation.

This paper was presented at International Conference on Nonlinear Operators, Differential Equations and Applications held in Cluj-Napoca (Romania) from August 24 to August 27, 2004.

where $h_1 \neq h_2,\ \beta, \gamma \neq 0,\ \lambda \in R$.

By a solution of equation (3) we mean a function $x \in C^1(R^2)$ which satisfy the relation (4) for any $t \in R,\ \lambda \in R$.

Remark. If $\lambda = 0$, the equation becomes a difference equation. Both the differential equation case $\lambda \neq 0$ and the difference equation case $\lambda = 0$, will be of interest.

In the first part of this section we assume that $\lambda \neq 0$. By the method of steps we'll find the solution for problem (3) + (4).

Let $t \in [t_0, t_0 + h_2]$. We have

$$-\lambda\varphi'(t,\lambda) = \alpha\varphi(t,\lambda) + \beta\varphi(t-h_1,\lambda) + \gamma x(t+h_2,\lambda) \tag{5}$$

It follows that

$$x(t) := x_1(t) = -\frac{1}{\gamma}[\alpha\varphi(t-h_2,\lambda) + \beta\varphi(t-h_1-h_2,\lambda) + \lambda\varphi\prime(t-h_2,\lambda)], \tag{6}$$

$t \in [t_0 + h_2, t_0 + 2h_2]$.

Let $t \in [t_0 + h_2, t_0 + 2h_2]$. We have

$$-\lambda x_1'(t,\lambda) = \alpha x_1(t,\lambda) + \beta y_1(t-h_1,\lambda) + \gamma x(t+h_2,\lambda) \tag{7}$$

where

$$y_1(t-h_1,\lambda) := \begin{cases} \varphi(t-h_1,\lambda), & h_1 > h_2 \\ \varphi(t-h_1,\lambda), & h_1 < h_2,\ t \in [t_0+h_2, t_0+h_1+h_2] \\ x_1(t-h_1,\lambda), & h_1 < h_2,\ t \in [t_0+h_1+h_2, t_0+2h_2] \end{cases} \tag{8}$$

It follows that

$$x(t) := x_2(t) = -\frac{1}{\gamma}[\alpha x_1(t-h_2,\lambda) + \beta y_1(t-h_1-h_2,\lambda) + \lambda x_1'(t-h_2,\lambda)], \tag{9}$$

$t \in [t_0 + 2h_2, t_0 + 3h_2]$.

The same way, it follows that

$$x_n(t) = -\frac{1}{\gamma}[\alpha x_{n-1}(t-h_2,\lambda) + \beta y_{n-1}(t-h_1-h_2,\lambda) + \lambda x_{n-1}'(t-h_2,\lambda)], \tag{10}$$

$t \in [t_0 + nh_2, t_0 + (n+1)h_2]$, where

$$y_{n-1}(t-h_1-h_2,\lambda) :=$$

$$\begin{cases} \varphi(t-h_1-h_2,\lambda), & h_1 > (n-1)h_2 \\ x_1(t-h_1-h_2,\lambda), & (n-1)h_2 > h_1 > (n-2)h_2, \\ \cdots & \\ x_{n-2}(t-h_1-h_2,\lambda), & 2h_2 > h_1 > h_2, \\ x_{n-2}(t-h_1-h_2,\lambda), & h_1 < h_2,\ t \in [t_0+nh_2, t_0+nh_2+h_1), \\ x_{n-1}(t-h_1-h_2,\lambda), & h_1 < h_2,\ t \in [t_0+nh_2+h_1, t_0+(n+1)h_2] \end{cases} \quad (11)$$

We apply the same method on the left of t_0 and it follows

$$x_{-n}(t) = -\frac{1}{\beta}[\alpha x_{-(n-1)}(t+h_1,\lambda)+\gamma y_{-(n-1)}(t+h_1+h_2,\lambda)+\lambda x'_{-(n-1)}(t+h_1,\lambda)], \quad (12)$$

$t \in [t_0 - (n+1)h_1, t_0 - nh_1]$, where

$$y_{-(n-1)}(t+h_1+h_2,\lambda) :=$$

$$\begin{cases} \varphi(t+h_1+h_2,\lambda), & h_2 > (n-1)h_1 \\ x_{-1}(t+h_1+h_2,\lambda), & (n-1)h_1 > h_2 > (n-2)h_1 \\ \cdots & \\ x_{-(n-2)}(t+h_1+h_2,\lambda), & 2h_1 > h_2 > h_1 \\ x_{-(n-2)}(t+h_1+h_2,\lambda), & h_2 < h_1,\ t \in [t_0-nh_1, t_0-nh_1+h_2) \\ x_{-(n-1)}(t+h_1+h_2,\lambda), & h_2 < h_1,\ t \in [t_0-nh_1+h_2, t_0+(n-1)h_1] \end{cases} \quad (13)$$

Next we have to find a condition for unique of the solution.

Let $\varphi \in C^\infty[t_0 - h_1, t_0 + h_2]$.

Let $x \in C^\infty(R)$ a solution of the problem (3)+(4).

We have

$$-\lambda x^{(k+1)}(t,\lambda) = \alpha x^{(k)}(t,\lambda)+\beta x^{(k)}(t-h_1,\lambda)+\gamma x^{(k)}(t+h_2,\lambda),\ k \in 0,1,\cdots,n. \quad (14)$$

For $t = t_0$ we have

$$-\lambda\varphi^{(k+1)}(t,\lambda) = \alpha\varphi^{(k)}(t_0,\lambda) + \beta\varphi^{(k)}(t_0-h_1,\lambda) + \gamma\varphi^{(k)}(t+h_2,\lambda),\ k \in 0,1,\cdots,n. \quad (15)$$

Theorem 1.1. *The problem (3)+(4) have a unique solution if the relation (15) is done and the solution have the form*

$$x(t,\lambda)=\begin{cases} x_{-k}(t,\lambda), & t\in[t_0-(k+1)h_1,t_0-kh_1],\ k=1,2,\cdots,n,\ n\to\infty \\ \varphi(t,\lambda), & t\in[t_0-h_1,t_0+h_2] \\ x_k(t,\lambda), & t\in[t_0+kh_2,t_0+(k+1)h_2] \end{cases} \quad (16)$$

where

$$x_k(t,\lambda)=-\frac{1}{\gamma}[\alpha x_{k-1}(t-h_2,\lambda)+\beta y_{k-1}(t-h_1-h_2,\lambda)+\lambda x'_{k-1}(t-h_2,\lambda)], \quad (17)$$

$t\in[t_0+kh_2,t_0+(k+1)h_2]$

$$y_{k-1}(t-h_1-h_2,\lambda):=$$

$$\begin{cases} \varphi(t-h_1-h_2,\lambda), & h_1>(k-1)h_2 \\ x_1(t-h_1-h_2,\lambda), & (k-1)h_2>h_1>(k-2)h_2, \\ \dots & \\ x_{k-2}(t-h_1-h_2,\lambda), & 2h_2>h_1>h_2, \\ x_{k-2}(t-h_1-h_2,\lambda), & h_1<h_2,\ t\in[t_0+kh_2,t_0+kh_2+h_1), \\ x_{k-1}(t-h_1-h_2,\lambda), & h_1<h_2, t\in[t_0+kh_2+h_1,t_0+(k+1)h_2] \end{cases} \quad (18)$$

$$x_{-k}(t,\lambda)=-\frac{1}{\beta}[\alpha x_{-(k-1)}(t+h_1,\lambda)+\gamma y_{-(k-1)}(t+h_1+h_2,\lambda)+\lambda x'_{-(k-1)}(t+h_1,\lambda)], \quad (19)$$

$t\in[t_0-(k+1)h_1,t_0-kh_1]$,

$$y_{-(k-1)}(t+h_1+h_2,\lambda):=$$

$$\begin{cases} \varphi(t+h_1+h_2,\lambda), & h_2>(k-1)h_1 \\ x_{-1}(t+h_1+h_2,\lambda), & (n-1)h_1>h_2>(n-2)h_1 \\ \dots & \\ x_{-(k-2)}(t+h_1+h_2,\lambda), & 2h_1>h_2>h_1 \\ x_{-(k-2)}(t+h_1+h_2,\lambda), & h_2<h_1,\ t\in[t_0-kh_1,t_0-kh_1+h_2) \\ x_{-(k-1)}(t+h_1+h_2,\lambda), & h_2<h_1,\ t\in[t_0-kh_1+h_2,t_0+(k-1)h_1] \end{cases} \quad (20)$$

Let now $\lambda = 0$. The problem becomes:

$$0 = \alpha x(t,0) + \beta x(t-h_1,0) + \gamma x(t+h_2,0),\ t \in R \tag{21}$$

$$x(t,0) = \varphi(t,0),\ t \in [t_0 - h_1, t_0 + h_2] \tag{22}$$

If we apply in the same way the method of steps it follows

$$x_{-k}(t,0) = -\frac{1}{\beta}[\alpha x_{-(k-1)}(t+h_1,0) + \gamma y_{-(k-1)}(t+h_1+h_2,0)], \tag{23}$$

$t \in [t_0 - (k+1)h_1, t_0 - kh_1]$,

$$y_{-(k-1)}(t+h_1+h_2,0) :=$$

$$\begin{cases} \varphi(t+h_1+h_2,0), & h_2 > (k-1)h_1 \\ x_{-1}(t+h_1+h_2,0), & (n-1)h_1 > h_2 > (n-2)h_1 \\ \cdots & \\ x_{-(k-2)}(t+h_1+h_2,0), & 2h_1 > h_2 > h_1 \\ x_{-(k-2)}(t+h_1+h_2,0), & h_2 < h_1,\ t \in [t_0 - kh_1, t_0 - kh_1 + h_2) \\ x_{-(k-1)}(t+h_1+h_2,0), & h_2 < h_1,\ t \in [t_0 - kh_1 + h_2, t_0 + (k-1)h_1] \end{cases} \tag{24}$$

$$x_k(t,0) = -\frac{1}{\gamma}[\alpha x_{k-1}(t-h_2,0) + \beta y_{k-1}(t-h_1-h_2,0)],\ t \in [t_0+kh_2, t_0+(k+1)h_2] \tag{25}$$

$$y_{k-1}(t-h_1-h_2,0) :=$$

$$\begin{cases} \varphi(t-h_1-h_2,0), & h_1 > (k-1)h_2 \\ x_1(t-h_1-h_2,0), & (k-1)h_2 > h_1 > (k-2)h_2, \\ \cdots & \\ x_{k-2}(t-h_1-h_2,0), & 2h_2 > h_1 > h_2, \\ x_{k-2}(t-h_1-h_2,0), & h_1 < h_2,\ t \in [t_0 + kh_2, t_0 + kh_2 + h_1), \\ x_{k-1}(t-h_1-h_2,0), & h_1 < h_2,\ t \in [t_0 + kh_2 + h_1, t_0 + (k+1)h_2] \end{cases} \tag{26}$$

and

$$x(t,0) = \begin{cases} x_{-k}(t,0), & t \in [t_0 - (k+1)h_1, t_0 - kh_1],\ k = 1, 2, \cdots, n,\ n \to \infty \\ \varphi(t,0), & t \in [t_0 - h_1, t_0 + h_2] \\ x_k(t,0), & t \in [t_0 + kh_2, t_0 + (k+1)h_2] \end{cases} \tag{27}$$

Theorem 1.2. *The problem (21) + (22) has a unique solution if and only if we have*

$$0 = \alpha\varphi^{(k)}(t_0, 0) + \beta\varphi^{(k)}(t_0 - h_1, 0) + \gamma\varphi^{(k)}(t_0 + h_2, 0),\ k \in 0, 1, \cdots, n. \tag{28}$$

We can give the following theorem

Theorem 1.3. *Let the problem (3) + (4) with the solution given by (16). If we put a limit on (16) when $\lambda \to 0$ we obtain the relation (27), which is the unique solution of the problem (21) + (22).*

2. **Main results**

In this section we consider the problem

$$-\lambda x\prime(t, \lambda) = f(t, x(t, \lambda), x(t - h_1, \lambda), x(t + h_2, \lambda)),\ t \in R \tag{29}$$

$$x(t, \lambda) = \varphi(t, \lambda),\ t \in [t_0 - h_1, t_0 + h_2],\ \varphi \in C^1[t_0 - h_1, t_0 + h_2] \tag{30}$$

where

$$\frac{\partial f(t, u)}{\partial u_j} \neq 0,\ j = 2, 3,\ u = (u_1, u_2, u_3),\ f \in C^\infty(R^4). \tag{31}$$

We first suppose that $\lambda \neq 0$.

Let the following conditions:

$(C1)$ For any $u_1,\ u_2,\ u_4,\ u_5 \in R$, there exist a unique $u_3 \in R$, $u_3 = f_1(u_1, u_2, u_4, u_5)$, $f_1 \in C^\infty(R^4)$, so that $u_5 = f(u_1, u_2, u_3, u_4)$.

$(C2)$ For any $u_1,\ u_2,\ u_3,\ u_5 \in R$, there exist a unique $u_4 \in R$, $u_4 = f_2(u_1, u_2, u_3, u_5)$, $f_2 \in C^\infty(R^4)$, so that $u_5 = f(u_1, u_2, u_3, u_4)$.

Remark. If $f \in C^\infty(R^4)$ and $x \in C^1(R^2)$ is a solution of the equation (29), then $x \in C^\infty(R^2)$.

Theorem 2.1. *Let $f \in C^\infty(R^4)$ satisfy the conditions $(C1)$, $(C2)$.*

If $\varphi \in C^\infty[t_0 - h_1, t_0 + h_2]$ then the problem (29) + (30) has a unique solution if and only if the following relation takes place:

$$-\lambda\varphi^{(k+1)}(t_0, \lambda) = [f(t, \varphi(t, \lambda), \varphi(t - h_1, \lambda), \varphi(t + h_2, \lambda))]^{(k)}_{t=t_0},\ k = 0, 1, \cdots, n. \tag{32}$$

Proof. By the method of steps we built the solution of the problem (29) + (30) as follows:

Let $t \in [t_0, t_0 + h_2]$,

$$-\lambda\varphi\prime(t,\lambda) = f(t, \phi(t,\lambda), \phi(t-h_1,\lambda), x(t+h_2,\lambda)). \tag{33}$$

From $(C2)$ we have

$$x(t,\lambda) := x_1(t,\lambda) = f_2(t-h_2, \varphi(t-h_2,\lambda), \varphi(t-h_1-h_2,\lambda), \lambda\varphi'(t-h_2,\lambda)), \tag{34}$$

$t \in [t_0 + h_2, t_0 + 2h_2]$.

By the same way we have:

$$x_k(t,\lambda) = f_2(t-h_2, x_{k-1}(t-h_2,\lambda), y_{k-1}(t-h_1-h_2,\lambda), \lambda x'_{k-1}(t-h_2,\lambda)), \tag{35}$$

$t \in [t_0 + kh_2, t_0 + (k+1)h_2]$, where

$$y_{k-1}(t-h_1-h_2,\lambda) :=$$

$$\begin{cases} \varphi(t-h_1-h_2,\lambda), & h_1 > (k-1)h_2 \\ x_1(t-h_1-h_2,\lambda), & (k-1)h_2 > h_1 > (k-2)h_2, \\ \dots & \\ x_{k-2}(t-h_1-h_2,\lambda), & 2h_2 > h_1 > h_2, \\ x_{k-2}(t-h_1-h_2,\lambda), & h_1 < h_2,\ t \in [t_0+kh_2, t_0+kh_2+h_1), \\ x_{k-1}(t-h_1-h_2,\lambda), & h_1 < h_2,\ t \in [t_0+kh_2+h_1, t_0+(k+1)h_2] \end{cases} \tag{36}$$

On the left of t_0 we obtain:

$$x_{-k}(t,\lambda) = f_1(t+h_1, x_{-(k-1)}(t+h_1,\lambda), y_{-(k-1)}(t+h_1+h_2,\lambda), \lambda x'_{-(k-1)}(t+h_1,\lambda)), \tag{37}$$

$t \in [t_0 - (k+1)h_1, t_0 - kh_1]$, where

$$y_{-(k-1)}(t+h_1+h_2,\lambda) :=$$

$$\begin{cases} \varphi(t+h_1+h_2,\lambda), & h_2 > (k-1)h_1 \\ x_{-1}(t+h_1+h_2,\lambda), & (n-1)h_1 > h_2 > (n-2)h_1 \\ \cdots & \\ x_{-(k-2)}(t+h_1+h_2,\lambda), & 2h_1 > h_2 > h_1 \\ x_{-(k-2)}(t+h_1+h_2,\lambda), & h_2 < h_1,\ t \in [t_0-kh_1, t_0-kh_1+h_2) \\ x_{-(k-1)}(t+h_1+h_2,\lambda), & h_2 < h_1,\ t \in [t_0-kh_1+h_2, t_0+(k-1)h_1] \end{cases} \tag{38}$$

For the regularity of the solution we have the conditions:

$$\begin{cases} \varphi(t_0+h_2,\lambda) = x_1(t_0+h_2,\lambda) \\ x_p(t_0+(p+1)h_2,\lambda) = x_{p+1}(t_0+(p+1)h_2,\lambda),\ p \geq 1 \\ \varphi(t_0-h_1,\lambda) = x_{-1}(t_0-h_1,\lambda) \\ x_{-p}(t_0-(p+1)h_1,\lambda) = x_{-(p+1)}(t_0-(p+1)h_1,\lambda),\ p \geq 1 \end{cases} \tag{39}$$

So the solution is

$$x(t,\lambda) = \begin{cases} x_{-k}(t,\lambda), & t \in [t_0-(k+1)h_1, t_0-kh_1],\ k = 1,2,\cdots,n,\ n \to \infty \\ \varphi(t,\lambda), & t \in [t_0-h_1, t_0+h_2] \\ x_k(t,\lambda), & t \in [t_0+kh_2, t_0+(k+1)h_2] \end{cases} \tag{40}$$

We have to prove the necessity of the (42). Let $x \in C^1(R)$ solution of the problem (29)+(30). Then $x \in C^\infty(R)$ is a solution. We have

$$-\lambda x^{(k+1)}(t,\lambda) = [f(t,x(t,\lambda),x(t-h_1,\lambda),x(t+h_2,\lambda))]^{(k)},\ t \in R,\ k = 0,1,\cdots,n \tag{41}$$

For $t = t_0$ it follows

$$-\lambda\varphi^{(k+1)}(t_0,\lambda) = [f(t,\varphi(t,\lambda),\varphi(t-h_1,\lambda),\varphi(t+h_2,\lambda))]^{(k)}_{t=t_0},\ k = 0,1,\cdots,n. \tag{42}$$

Let now $\lambda = 0$

The problem becomes

$$0 = f(t,x(t,0),x(t-h_1,0),x(t+h_2,0)),\ t \in R \tag{43}$$

$$x(t,0) = \varphi(t,0),\ t \in [t_0-h_1,t_0+h_2],\ \varphi \in C^1[t_0-h_1,t_0+h_2] \tag{44}$$

Theorem 2.2. *Let $f \in C^{\infty}(R^4)$ satisfy the conditions $(C1)$, $(C2)$.*

If $\varphi \in C^{\infty}[t_0 - h_1, t_0 + h_2]$ then the problem (43) + (44) has a unique solution if and only if the following relation takes place:

$$-\lambda \varphi^{(k+1)}(t_0, 0) = [f(t, \varphi(t,0), \varphi(t-h_1, 0), \varphi(t+h_2, 0))]^{(k)}_{t=t_0}, \; k = 0, 1, \cdots, n. \quad (45)$$

Proof. The proof is similar with the proof of the last theorem. We find by the method of steps that the unique solution of the problem (43) + (44) has the form

$$x(t,0) = \begin{cases} x_{-k}(t,0), & t \in [t_0 - (k+1)h_1, t_0 - kh_1], \; k = 1, 2, \cdots, n, \; n \to \infty \\ \varphi(t,0), & t \in [t_0 - h_1, t_0 + h_2] \\ x_k(t,0), & t \in [t_0 + kh_2, t_0 + (k+1)h_2] \end{cases} \quad (46)$$

where

$$x_k(t,0) = f_2(t - h_2, x_{k-1}(t - h_2, 0), y_{k-1}(t - h_1 - h_2, 0)), t \in [t_0 + kh_2, t_0 + (k+1)h_2], \quad (47)$$

$$y_{k-1}(t - h_1 - h_2, 0) :=$$

$$\begin{cases} \varphi(t - h_1 - h_2, 0), & h_1 > (k-1)h_2 \\ x_1(t - h_1 - h_2, 0), & (k-1)h_2 > h_1 > (k-2)h_2, \\ \dots & \\ x_{k-2}(t - h_1 - h_2, 0), & 2h_2 > h_1 > h_2, \\ x_{k-2}(t - h_1 - h_2, 0), & h_1 < h_2, \; t \in [t_0 + kh_2, t_0 + kh_2 + h_1), \\ x_{k-1}(t - h_1 - h_2, 0), & h_1 < h_2, \; t \in [t_0 + kh_2 + h_1, t_0 + (k+1)h_2] \end{cases} \quad (48)$$

$$x_{-k}(t,0) = f_1(t + h_1, x_{-(k-1)}(t + h_1, 0), y_{-(k-1)}(t + h_1 + h_2, 0)), \quad (49)$$

$t \in [t_0 - (k+1)h_1, t_0 - kh_1]$,

$$y_{-(k-1)}(t + h_1 + h_2, 0) :=$$

$$\begin{cases} \varphi(t+h_1+h_2,0), & h_2>(k-1)h_1 \\ x_{-1}(t+h_1+h_2,0), & (n-1)h_1>h_2>(n-2)h_1 \\ \dots & \\ x_{-(k-2)}(t+h_1+h_2,0), & 2h_1>h_2>h_1 \\ x_{-(k-2)}(t+h_1+h_2,0), & h_2<h_1,\ t\in[t_0-kh_1,t_0-kh_1+h_2) \\ x_{-(k-1)}(t+h_1+h_2,0), & h_2<h_1,\ t\in[t_0-kh_1+h_2,t_0+(k-1)h_1] \end{cases} \tag{50}$$

It is natural that the solution of problem (29) + (30) converge, when $\lambda \to 0$, to the solution of the problem (43) + (44).

So we can give the following theorem:

Theorem 2.3. *Let $f \in C(R^4), \lambda$ parameter. Let $x(t,\lambda)$ given by (40) the solution of the problem (29) + (30). If we put $\lambda \to 0$ then $x(t,\lambda)$ given by (40) converge punctually to $x(t,0)$ given by (46).*

3. **The comparing of the solutions**

Let the equation

$$-\lambda x\prime(t,\lambda) = f(t, x(t,\lambda), x(t-h_1,\lambda), x(t+h_2,\lambda)),\ t \in R \tag{51}$$

where

(i) $h_1 \neq h_2$;

(ii) $f : R^4 \to R,\ f(t,u)$ - continuous, local Lipschitz on u, $u = (u_1, u_2, u_3)$;

(iii) for any $t \in R$: $\frac{\partial f(t,u)}{\partial u_j} \geq 0$, $u \in R^3$, $j = 2, 3$, $u = (u_1, u_2, u_3)$.

Lemma 3.1. *Assume that (ii) and (iii) above hold. Let $x_j : R \to R$, for $j = 1, 2$, be two solutions of equation (51) at some nonzero parameter value $\lambda \in R^*$. Assume that*

$$x_1(t,\lambda) \geq x_2(t,\lambda),\ t \in R. \tag{52}$$

Then if $x_1(\tau,\lambda) = x_2(\tau,\lambda)$ at some $\tau \in R$, we have that

$x_1(\xi,\lambda) = x_2(\xi,\lambda)$ for all $\xi \geq \tau$, in case $\lambda > 0$, and that

$x_1(\xi,\lambda) = x_2(\xi,\lambda)$ for all $\xi \leq \tau$, in case $\lambda < 0$.

Proof. Let $y(t,\lambda) := x_1(t,\lambda) - x_2(t,\lambda) \geq 0$, and we assume that the inequality (52) is an equality at some $\tau \in R$.

It means that $y(\tau,\lambda) = 0$, so that $x_1(\tau,\lambda) = x_2(\tau,\lambda)$.

We prove for $c > 0$; the proof when $c < 0$ being similar.

Let the function

$$H(\xi,u) := -\lambda^{-1}[G(\xi, u + x_2(\xi,\lambda), x_1(\xi - h_1,\lambda), x_1(\xi + h_2,\lambda)) \\ -G(\xi, x_2(\xi,\lambda), x_2(\xi - h_1,\lambda), x_2(\xi + h_2,\lambda)]. \tag{53}$$

We observe that $u = y(\xi,\lambda)$ satisfies $u' = H(\xi,u)$.

From (52) and (iii) we have that $H(\xi,0) \leq 0$, for every $\xi \in R$.

It follows that $y(\xi,\lambda) \leq 0$ for all $\xi \geq \tau$ by a standard differential inequality. So that $y(\xi,\lambda) = 0$ for all $\xi \geq \tau$.

Thus $x_1(\xi,\lambda) = x_2(\xi,\lambda)$ for all $\xi \geq \tau$, in case $\lambda > 0$ and $x_1(\xi,\lambda) = x_2(\xi,\lambda)$ for all $\xi \leq \tau$, in case $\lambda < 0$.

Lemma 3.2. *Assume that the conditions of lemma (3.1) hold, except that solutions x_1, x_2 satisfy equation (51) at different values of λ, with $\lambda_1 > \lambda_2$, and where either $\lambda_1 = 0$ or $\lambda_2 = 0$ are permitted. Suppose that $x_1(\tau,\lambda_1) = x_2(\tau,\lambda_2)$ at some $\tau \in R$. Then if*

$$x_2(\xi,\lambda_2) \text{ is monotone increasing in } \xi \in R$$

and $\lambda_1 > 0$ we have that $x_1(\xi,\lambda_1) = x_2(\xi,\lambda_2)$ is constant for all $\xi \geq \tau + h_1$; while if

$$x_1(\xi,\lambda_1) \text{ is monotone increasing in } \xi \in R$$

and $\lambda_2 > 0$ we have that $x_1(\xi,\lambda_1) = x_2(\xi,\lambda_2)$ is constant for all $\xi \leq \tau - h_2$;

Proof. The proof is very similar to that of Lemma (3.1) except for the choice of the function H. Several cases must be considered, based on the signs of λ_1 and λ_2.

First, we suppose that (3.2) holds and $\lambda_2 \neq 0$.

Let

$$H(\xi,u) := -\lambda_1^{-1}G(\xi, u + x_2(\xi,\lambda), x_1(\xi - h_1,\lambda), x_1(\xi + h_2,\lambda)) \\ +\lambda_2^{-1}G(\xi, x_2(\xi\lambda), x_2(\xi - h_1,\lambda), x_2(\xi + h_2,\lambda). \tag{54}$$

Then $u = x_1(\xi, \lambda_1) - x_2(\xi, \lambda_1)$ satisfy $u' = H(\xi, u)$.

By replacing x_1 with x_2 in the formula of H and using the inequality (52) and (iii), form (3.2) follows that

$H(\xi, 0) \leq (\lambda_2\lambda_1^{-1} - 1)x_2'(\xi, \lambda_2) \leq 0.$

Thus $x_1(\xi, \lambda_1) - x_2(\xi, \lambda_2) \leq 0$ for all $\xi \geq \tau$.

So that $x_1(\xi, \lambda_1) = x_2(\xi, \lambda_2)$, $\xi \geq \tau$.

From (51) we have

$\lambda_1 x_1'(\xi, \lambda_1) = \lambda_2 x_2'(\xi, \lambda_2)$ for all $\xi \geq \tau + h_1$.

If $\lambda_1 \neq \lambda_2$ we have $x_1'(\xi, \lambda_1) = x_2'(\xi, \lambda_2) = 0$, $\xi \geq \tau + h_1$; so $x_1(\xi, \lambda_1) = x_2(\xi, \lambda_2) = const$, $\xi \geq \tau + h_1$.

Now suppose that (3.2) holds and $\lambda_2 = 0$.

Let

$H(\xi, u) := -\lambda_1^{-1} G(\xi, u, x_1(\xi - h_1, \lambda_1), x_1(\xi + h_2, \lambda_1))$

$$\widetilde{H}(\xi, u) = \begin{cases} H(\xi, x_2(\xi, \lambda_2)), & u \geq x_2(\xi, \lambda_2) \\ H(\xi, u), & u \leq x_2(\xi, \lambda_2) \end{cases} \tag{55}$$

H can be easily checks that $H(\xi, x_2(\xi, \lambda_2)) \leq 0$, so that $\widetilde{H}(\xi, u) \leq 0$. From inequality, form the fact that x_2 is monotone increasing and that $\widetilde{H}$ satisfies the standard Caratheodory and Lipschitz conditions, we have that the unique solution $u = x_3(\xi, \lambda_2)$ satisfies $u' = H(\xi, u)$, for any $\xi \geq \tau$.

But $u = x_1(\xi, \lambda_1)$ satisfies $u' = H(\xi, u)$ and $x_3(\tau) = x_2(\tau) = x_1(\tau)$.

So that $x_1(\xi, \lambda_1) \leq x_2(\xi, \lambda_2)$ for any $\xi \geq \tau$.

Thus $x_1(\xi, \lambda_1) = x_2(\xi, \lambda_2)$ for any $\xi \geq \tau$.

So $x_1'(\xi) = x_2'(\xi) = 0$ for any $\xi \geq \tau + h_1$.

It follows that $x_1(\xi, \lambda_1) = x_2(\xi, \lambda_2) = const$ for any $\xi \geq \tau + h_1$.

References

[1] Darzu, V. A., *Some Existence Results for Differential Equations of Mixed Type,* Itinerant Seminar Tiberiu Popoviciu, mai 2001, 43-46.

[2] Darzu, V. A., *Weeler-Feynman problem on a compact interval*, Semin. Fixed Point Theory Cluj-Napoca 3(2002), 389-392.

[3] Darzu, V.A., *Functional Differential Equation of Mixed Type, via Weakly Picard Operators*, Proc. 6th Conf. of the Romanian Math. Soc., 276-284.

[4] Darzu, V. A., *Data dependence for functionall differential equations of mixed type*, Mathematica (Cluj) (to appear).

[5] Wheeler, J. A., Feynman, R. P., *Classical electordynamics in terms of direct interparticles action*, Rev. Mod. Phys. 21(1949), 425-433.

[6] Mallet-Paret, J., *The global structure of travelling waves in spatially discrete dynamical systems*, J. Dynam. Diff. Eq., vol. 11, no. 1(1999), 49-127.

[7] Mallet-Paret, J., *Exponential dichotonomies and Wiener-Hopf factorizations for mixed-type functional differential equations*, 2001, 1-86.

[8] Mallet-Paret, J., *The Fredholm alternative for functional differential equations of mixed type*, J. Dynam. Diff. Eq., vol 11, no.1(1999), 1-46.

[9] Precup, R., *Some Existence Results for Differential Equations with Both Retarded and Advanced Arguments*, Mathematica, Cluj-Napoca, 2002, to appear.

[10] Rus, I. A., *Ecuaţii diferenţiale. Ecuaţii integrale şi sisteme dinamice*, Transilvnia Press, Cluj-Napoca, 1996.

[11] Rus, I. A., Iancu, C., *Wheeler-Feynman problem for mixed first order functional differential equations*, Seminar Itinerant Tiberiu Popoviciu, Cluj-Napoca, mai 2000, 197-200.

[12] Rus, I. A., *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.

Department of Applied Mathematics, Babeş-Bolyai University,
Str. M. Kogălniceanu 1, Cluj-Napoca, Romania
E-mail address: vdarzu@math.ubbcluj.ro