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ON THE EXISTENCE OF VIABLE SOLUTIONS FOR A CLASS OF NONAUTONOMOUS NONCONVEX DIFFERENTIAL INCLUSIONS

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Abstract. We prove the existence of viable solutions to the Cauchy problem $x' \in F(t,x), x(0) = x_0$ in M, where F is a multifunction and M is a convex locally compact set of a Hilbert space that satisfy $F(t,x) \cap K_x M \cap \partial V(x) \neq \emptyset$, with $K_x M$ the contingent cone to M at x and ∂V is the subdifferential of a convex function V.

1. **Introduction**

Consider H a real Hilbert space and $F : M \subset H \to \mathcal{P}(H)$ a multifunction that defines the Cauchy problem

$$x' \in F(x), \quad x(0) = x_0, \tag{1.1}$$

In the theory of differential inclusions the viability problem consists in proving the existence of viable solutions, i.e. $\forall t$, $x(t) \in M$, to the Cauchy problem (1.1).

Under the assumptions that $H = R^n$, F is an upper semicontinuous nonempty convex compact valued multifunction and M is locally compact, in [5] Haddad proved that a necessary and sufficient condition for the existence of viable trajectories starting from $x_0 \in M$ of problem (1.1) is the tangential condition

$$\forall x \in M \quad F(x) \cap K_x M \neq \emptyset, \tag{1.2}$$

where $K_x M$ is the contingent cone to M at $x \in M$.

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Rossi, in [7], proved the existence of viable solutions to problem (1.1) replacing the convexity conditions on the images on F with

$$F(x) \subset \partial V(x) \quad \forall x \in M, \tag{1.3}$$

where ∂V is the subdifferential, in the sense of Convex Analysis, of a proper convex function V. In [4] condition (1.3) is improved in the sense that instead of (1.2) and (1.3) we assume that $F(.)$ verifies

$$F(x) \cap K_x M \cap \partial V(x) \neq \emptyset \quad \forall x \in M, \tag{1.4}$$

with V as in [7], provided M is convex.

The aim of the present paper is to extend the result in [4] to the case of nonautonomous problems

$$x' \in F(t,x), \quad x(0) = x_0. \tag{1.5}$$

We note that in [6] a similar type of result is proved for a function V that is assumed to be lower regular, i.e. a locally Lipschitz continuous function whose upper Dini directional derivative coincides with the Clarke directional derivative.

The idea of the proof of our result is to use the regularizing technique in [6] and to apply the known result for autonomous problems in [4].

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2. **Preliminaries**

Let H be a real separable Hilbert space and $\Omega \subset H$ a given set. By $\mathcal{P}(H)$ we denote the family of all subsets of H. A multifunction $F : \Omega \to \mathcal{P}(H)$ is called (Hausdorff) upper semicontinuous at $x_0 \in \Omega$, $\forall \epsilon > 0$ there exists $\delta > 0$ such that $||x - x_0|| < \delta$ implies $F(x) \subset F(x_0) + \epsilon B$, where B is the unit ball in H. For $\epsilon > 0$ we put $B(x, \epsilon) = \{y \in H; ||y - x|| < \epsilon\}$.

Let $V : H \to R \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in H; V(x) < +\infty\}$. If $D(V) \neq \emptyset$, then f is called *proper*. We recall that the *subdifferential* (in the sense of Convex Analysis) of the convex function V is the multifunction $\partial V : H \to \mathcal{P}(H)$ defined by

$$\partial V(x) = \{y \in H; \quad V(z) - V(x) \geq < y, z - x > \quad \forall z \in H\}.$$

In what follows we assume:

Hypothesis 2.1. i) $F : [0, \infty) \times M \subset H \to \mathcal{P}(H)$ is a bounded set valued map, measurable in t, upper semicontinuous with respect to x, with nonempty closed values.

ii) There exists a proper lower semicontinuous convex function $V : H \to R \cup \{+\infty\}$ such that

$$F(t, x) \cap K_x M \cap \partial V(x) \neq \emptyset \quad \forall x \in M, \ a.e. \ t \in [0, \infty), \tag{2.1}$$

where $K_x M = \{v \in H; \quad \liminf_{h \to 0+} \frac{1}{h} d(x + hv, M) = 0\}$ is the contingent cone to M at $x \in M$.

3. The main result

Our main result is the following.

Theorem 3.1. *Let $M \subset H$ be a convex and locally compact set and let $F : [0, \infty) \times M \subset H \to \mathcal{P}(H)$ be a set-valued map satisfying Hypothesis 2.1.*

Then for every $x_0 \in M$ there exists $T > 0$ such that problem (1.5) admits a solution on $[0, T]$ satisfying $x(t) \in M$, $\forall t \in [0, T]$.

Proof. Let $x_0 \in M$. Since $M \subset H$ is locally compact, there exists $r > 0$ such that $M_0 := M \cap B(x_0, r)$ is compact. Consider $L := \sup_{(t,x) \in [0,\infty) \times M} ||F(t, x)||$, define $T := \frac{r}{L+1}$ and take $n \in N$ such that $\frac{1}{n} < T$.

By regularizing the set valued map F on the right hand side of the Cauchy problem (1.5) we reduce the nonautonomous problem to the autonomous case ([6]). We can find a countable collection of disjoint subintervals $(a_j, b_j) \subset [0, T]$, $j = 1, 2, ...$ such that their total length is less then $\frac{1}{n}$ and a set valued map F_n defined on $D := ([0, T] \setminus \cup_{j=1}^{\infty} (a_j, b_j)) \times M$ that is jointly upper semicontinuous and $F_n(t, x) \subset F(t, x)$

for each $(t,x) \in D$. Moreover, if $u(.)$ and $v(.)$ are measurable functions on $[0,T]$ such that $u(t) \in F(t,v(t))$ a.e. $t \in [0,T]$ then for a.e. $t \in ([0,T]\setminus \cup_{j=1}^{\infty} (a_j,b_j))$ we have $u(t) \in F_n(t,v(t))$ (we refer to [8] for this Scorza Dragoni type theorem). It is obvious that all trajectories of F are also trajectories of F_n. We extend F_n to the whole $[0,T] \times M$. We define

$$\tilde{F}_n(t,x) = \begin{cases} F_n(t,x) & \text{if} \quad t \in [0,T]\setminus \cup_{j=1}^{\infty} (a_j,b_j) \\ F_n(a_j,x) & \text{if} \quad a_j < t < \frac{a_j+b_j}{2} \\ F_n(b_j,x) & \text{if} \quad \frac{a_j+b_j}{2} < t < b_j \\ F_n(a_j,x) \cup F_n(b_j,x) & \text{if} \quad t = \frac{a_j+b_j}{2}. \end{cases}$$

It is easy to see that $\tilde{F}_n(.,.)$ still satisfies the tangential condition (2.1). On the other hand, according to Lemma 4 in [6], $\tilde{F}_n(.,.)$ is upper semicontinuous on $[0,T] \times M$.

By extending the state space from H to $R \times H$ we can reduce our problem to the autonomous case. For every $(t,x) \in [0,T] \times M$ we define

$$\tilde{V}(t,x) = t + V(x).$$

Obviously, $\tilde{V}(.,.)$ is a proper lower semicontinuous convex function and $(1,v) \in \partial\tilde{V}(t,x)$ if and only if $v \in \partial V(x)$ for all $(t,x) \in [0,T] \times M$. At the same time, standard arguments show that $(1,v) \in K_{(t,x)}([0,T] \times M)$ if and only if $v \in K_x M$.

Therefore, the tangential condition (2.1) implies that

$$(1, \tilde{F}_n(t,x)) \cap K_{(t,x)}([0,T] \times M) \cap \partial\tilde{V}(t,x) \neq \emptyset \quad \forall (t,x) \in [0,T] \times M. \tag{3.1}$$

Thus applying Theorem 3.1 in [4] we obtain the existence of an absolutely continuous function $x_n(.) : [0,T] \to H$ that satisfies

$$(1, x_n'(t)) \in (1, \tilde{F}_n(t, x_n(t))) \cap \partial\tilde{V}(t, x_n(t)) \quad a.e.\ [0,T], \quad x_n(0) = x_0$$

and

$$(t, x_n(t)) \in [0,T] \times M \quad \forall t \in [0,T].$$

It follows that $x_n(.)$ verifies

$$x_n'(t) \in F_n(t, x_n(t)) \cap \partial V(x_n(t)) \quad a.e.\ [0,T], \quad x_n(0) = x_0 \tag{3.2}$$

and

$$x_n(t) \in M \quad \forall t \in [0,T]. \tag{3.3}$$

Therefore from (3.2) we have

$$||x_n'(t)|| \leq L. \tag{3.4}$$

On the other hand, from (3.3) $graph(x_n(.))$ is contained in $[0,T] \times M$ and $x_n(.)$ is also a solution to the inclusion (1.5) except for a set (say) E_n of measure not exceeding $\frac{1}{n}$ for each $n \in N$. Hence, from (3.4) and Theorem III. 27 in [3] there exists a subsequence (again denoted by $x_n(.)$) and an absolutely continuous function $x(.) : [0,T] \to H$ such that

$$x_n(.) \quad \text{converges uniformly to} \quad x(.),$$

$$x_n'(.) \quad \text{converges weakly in } L^2([0,T], H) \text{ to} \quad x'(.).$$

Since $V(.)$ is lower semicontinuous, it follows that $graph(\partial V)$ is closed and thus, by (3.2), one has

$$x'(t) \in \partial V(x(t)) \quad a.e.\ [0,T]. \tag{3.5}$$

We apply Lemma 3.3 in [2] and by (3.5) we obtain

$$(V(x(t)))' = < x'(t), x'(t) > = ||x'(t)||^2 \quad a.e.\ [0,T];$$

and thus, $V(x(T)) - V(x_0) = \int_0^T ||x'(t)||^2 dt$.

On the other hand, from (3.2) we deduce that

$$\int_0^T ||x_n'(t)||^2 dt = \int_0^T (V \circ x_n)'(t) dt = V(x_n(T)) - V(x_0).$$

Hence, by the lower semicontinuity of V, we get

$$\lim_{n\to\infty} \int_0^T ||x_n'(t)||^2 dt = V(x(T)) - V(x_0) = \int_0^T ||x'(t)||^2 dt$$

and so $\{x_n'(.)\}$ converges strongly in $L^2([0,T], H)$.

Hence, there exists a subsequence (still denoted) $x'_n(.)$ which converges pointwise almost everywhere to $x'(.)$. From (3.2) and the fact that $graph(F)$ is closed we have

$$x'(t) \in F(t, x(t)) \quad a.e. \ [0, T]$$

and from (3.3) we obtain that $\forall t \in [0, T]$, $x(t) \in M$.

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