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QUANTITATIVE APPROXIMATIONS BY USING SCALING TYPE FUNCTIONS

OCTAVIAN AGRATINI

Abstract. The focus of the paper is to study a class of linear positive operators constructed by using a quasi-scaling type function. Jackson type inequalities are established in the framework of different function spaces.

1. **Introduction**

In Approximation Theory an interesting tool with a rich mathematical content and great potential for applications, is materialized by sequences of linear positive operators generated by a scaling type function.

The aim of the present note is to investigate a general class $(L_k)_{k\in\mathbb{Z}}$ of linear positive operators of wavelet type. Our paper is designed as follows. Following [1], in Section 2 we recall the construction of L_k, $k \in \mathbb{Z}$, operators and we indicate the main notations and results which will be used in the sequel. Further on, in Section 3 we use this class to approximate smooth real valued signals, more precisely, functions which possess derivatives of high order. We establish both pointwise and global estimates of the rate of convergence of our operators. Under additional assumptions, we prove that each $\gamma^{-2}\delta^{-1}L_k$ operator has the degree of exactness equal to 1. The last section is devoted to estimate the approximation of bounded functions by L_k, with the help of a Lipschitz-type maximal function.

Clearly, the research along a certain line can be developed by different angles. We point out that our approach is made by using tools and methods which characterize

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the approximation of functions by linear positive operators. Similar results are quite familiar in the Littlewood-Paley and wavelets literature. A good illustration of this can be found in Daubechies' book [4; *Section 6.5*] and elsewhere.

2. Background and preliminaries

Setting $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$, we consider a bi-dimensional net $(a_k, \delta j)$, $(k,j) \in \mathbb{Z}\times\mathbb{Z}$, $\delta \in]0,\infty[$, and

$$a_{-k} = a_k^{-1},\ 0 < a_k < a_{k+1},\ \text{for every } k \in \mathbb{N}_0. \tag{1}$$

Clearly, $a_{-(p+1)} < a_{-p}$ for each $p \in \mathbb{N}_0$ and $a_0 = 1$. We point out that the above net generalizes the couples $(2^k, j)$, $(k,j) \in \mathbb{Z}\times\mathbb{Z}$, broad used in the construction of many wavelet type discrete operators. Operating both on the sequence $(a_k)_{k\in\mathbb{N}_0}$ and on the ratio δ, we are able to transform the net in accordance with the problem data and, therefore, it is more flexible then the previous one.

Let $L_{1,loc}(\mathbb{R})$ be the vector space of the real-valued functions defined on $\mathbb{R}$ and locally integrable, i.e. integrable on any compact interval of the real line. We make the following informal definition.

Definition 2.1. *Let $\delta > 0$ be fixed. A function $\varphi : \mathbb{R} \to [0,\infty[$ satisfying the following conditions:*

(i) φ is a bounded function belonging to $L_{1,loc}(\mathbb{R})$,

(ii) a positive constant α exists such that $\operatorname{supp}(\varphi) \subset [-\alpha, \alpha]$, (2)

(iii) a positive constant γ exists with the property

$$\sum_{j=-\infty}^{\infty} \varphi(x + \delta j) = \gamma,\ \textit{for every } x \in \mathbb{R}, \tag{3}$$

is called a scaling function of (δ, γ) type.

Using the sequence $(a_k)_{k\in\mathbb{Z}}$ defined by (1) and a scaling function φ of (δ,γ) type we generate the functions

$$\varphi_{k,j}(x) := \sqrt{a_k}\,\varphi(a_k x + \delta j), \quad x \in \mathbb{R},\ (k,j) \in \mathbb{Z}\times\mathbb{Z}. \tag{4}$$

As usual in wavelet transforms, k is named the dilation index and j is named the translation index. Dilation by larger k compresses the function φ on the x-axis.

Altering j has the effect of sliding the function φ along the x-axis. We mention that condition (3) has nothing to do with the property of an orthogonal scaling function of a multiresolution analysis (MRA), intensively used in the signals theory.

At this point we are in position to introduce the announced sequence of operators.

For every $k \in \mathbb{Z}$ and $f \in L_{1,loc}(\mathbb{R})$ we define the operator L_k as follows

$$(L_k f)(x) := \sum_{j=-\infty}^{\infty} (f, \varphi_{k,j}) \varphi_{k,j}(x), \quad x \in \mathbb{R}, \tag{5}$$

where the functions $\varphi_{k,j}$ are given by (4) and $(f, \varphi_{k,j}) = \int_{\mathbb{R}} f(t)\varphi_{k,j}(t)dt$.

As usual, we denote by $C(\mathbb{R})$ ($B(\mathbb{R})$, respectively) the space of all continuous (bounded, respectively) real valued functions on $\mathbb{R}$. The spaces $B(\mathbb{R})$ and $B(\mathbb{R}) \cap C(\mathbb{R})$ can be equipped with the norm $\|\cdot\|_\infty$ of the uniform convergence (briefly, the sup-norm). Also $L_p(\mathbb{R})$, $p \geq 1$, stands for the vector space of all real valued Lebesgue integrable functions defined on $\mathbb{R}$ endowed with the usual norm $\|\cdot\|_{L_p(\mathbb{R})}$. In the Hilbert space of square integrable functions, the inner product is denoted by $(\cdot, \cdot)$.

Examining Definition 2.1 we deduce that φ belongs to the Lebesgue space $L_2(\mathbb{R})$. The same statement is true for $\varphi_{k,j}$. Also, for each $(k, j) \in \mathbb{Z} \times \mathbb{Z}$ the coefficient $(f, \varphi_{k,j})$ exists and is finite. Because of the function φ has bounded support, for any real x the summation in (5) involves only a finite number of terms and, consequently, $(L_k f)(x)$ is well-defined on $\mathbb{R}$.

A more explicit look of $L_k f$ is the following

$$(L_k f)(x) = \sum_{j=-\infty}^{\infty} \varphi(a_k x + \delta j) \int\limits_{\mathrm{supp}(\varphi)} \varphi(u) f\left(\frac{u - \delta j}{a_k}\right) du. \tag{6}$$

The construction of $L_k f$ guarantees that L_k is a positive linear operator.

In the particular case $a_k = 2^k$ this operator becomes the operator A_k studied in [3]. The authors have used a scaling function φ of (1,1) type.

As regards L_k operator, a result presented in [1; *Theorem 1*] will be read as follows.

Proposition 2.1. *Let L_k, $k \in \mathbb{Z}$, be defined by (5). For every function $f \in C(\mathbb{R})$ the following inequality*

$$|(L_k f)(x) - \gamma^2 \delta f(x)| \leq \gamma^2 \delta \omega(f; 2\alpha a_{-k}), \quad k \in \mathbb{Z},\ x \in \mathbb{R},$$

holds true, where α is given at (2) and $\omega(f;\cdot)$ represents the modulus of continuity associated to f.

Further on we collect some direct properties of the functions φ and $\varphi_{k,j}$.

Lemma 2.2. *If φ is a scaling function of (δ, γ) type then one has*

(i) $\|\varphi\|_{L_1(\mathbb{R})} = \int_{\mathbb{R}} \varphi(x)dx = \int_{\mathbb{R}} \varphi(x + \delta j)dx = \gamma\delta,\ j \in \mathbb{Z};$ (7)

(ii) $\dfrac{\gamma\delta}{\sqrt{2\alpha}} \leq \|\varphi\|_{L_2(\mathbb{R})} \leq \sqrt{\gamma\delta \sup_{x\in\mathbb{R}} \varphi(x)};$

(iii) $\|\varphi_{k,j}\|_{L_1(\mathbb{R})} = \sqrt{a_{-k}}\gamma\delta,\ \|\varphi_{k,j}\|_{L_2(\mathbb{R})} = \|\varphi\|_{L_2(\mathbb{R})}.$

Since the proof is based on simple computations, we omit it.

We end this section proving that L_k enjoys the self-adjointness property on the Hilbert space $L_2(\mathbb{R})$.

Lemma 2.3. *For every f and g belonging to $L_2(\mathbb{R})$, the operator L_k defined by (5) verifies $(L_k f, g) = (f, L_k g)$.*

Proof. We can write successively

$$\begin{aligned}(L_k f, g) &= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \varphi_{k,j}(t) f(t) dt \Big) \varphi_{k,j}(x) g(x) dx \\ &= \sum_{j\in\mathbb{Z}} \Big(\int_{\mathbb{R}} \varphi_{k,j}(t) f(t) dt \Big) (g, \varphi_{k,j}) = (f, L_k g).\end{aligned}$$

□

3. **Estimates for high order differentiable functions**

In most practical problems, the functions possess some degree of smoothness.

Letting $C^n(\mathbb{R})$, $n \in \mathbb{N}$, the space of n-times continuously differentiable real valued functions defined on $\mathbb{R}$, we are concerned to give bounds for the approximation error $|L_k f - f|$, where $f \in C^n(\mathbb{R})$.

At first step we recall the Taylor formula. If $f \in C^n(\mathbb{R})$ then the following identity

$$f(y) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!}(y-x)^i + \frac{1}{(n-1)!}\int_x^y (f^{(n)}(t) - f^{(n)}(x))(y-t)^{n-1}dt \tag{8}$$

holds true for every $(x,y) \in \mathbb{R}\times\mathbb{R}$.

At second step we need a technical result useful in the proof of Theorem 3.2.

Lemma 3.1. *Let φ be a scaling function of (δ,γ) type. Let fix $(x,a_k) \in \mathbb{R}\times]0,\infty[$ and define*

$$\begin{cases} J_{k,x} := \{j \in \mathbb{Z} |\ a_k x + j\delta \in [-\alpha,\alpha]\}, \\ r_n(f;a_k,x,t) := \left|\int_x^{t/a_k} (f^{(n)}(u) - f^{(n)}(x))\left(\frac{t}{a_k} - u\right)^{n-1} du\right|,\ f \in C^n(\mathbb{R}) \end{cases} \tag{9}$$

For each $j_0 \in J_{k,x}$ and each $t \in [-\alpha - j_0\delta, \alpha - j_0\delta]$, the following inequalities hold

$$(i)\ \left|\frac{t}{a_k} - x\right| \le \frac{2\alpha}{a_k}, \quad (ii)\ r_n(f;a_k,x,t) \le \frac{1}{n}\left(\frac{2\alpha}{a_k}\right)^n \omega\left(f^{(n)};\frac{2\alpha}{a_k}\right). \tag{10}$$

Proof. Since $-2\alpha + a_k x \le -\alpha - j_0\delta \le t \le \alpha - j_0\delta \le 2\alpha + a_k x$, the first relation is evident. In order to prove the second inequality, we shall analyze 2 cases taking in view the first inequality of this Lemma.

Case 1. $x < t/a_k$. We have

$$\begin{aligned} r_n(f;a_k,x,t) &\le \int_x^{t/a_k} |f^{(n)}(u) - f^{(n)}(x)|\left(\frac{t}{a_k} - u\right)^{n-1} du \\ &\le \int_x^{t/a_k} \omega(f^{(n)};|u-x|)\left(\frac{t}{a_k} - u\right)^{n-1} du \\ &\le \int_x^{t/a_k} \left(\frac{t}{a_k} - u\right)^{n-1} du\, \omega\left(f^{(n)};\left|\frac{t}{a_k} - x\right|\right) \\ &\le \frac{1}{n}\left(\frac{t}{a_k} - x\right)^n \omega\left(f^{(n)};\frac{2\alpha}{a_k}\right). \end{aligned}$$

Case 2. $x > t/a_k$. Following the same line, we get

$$r_n(f;a_k,x,t) \le \int_{t/a_k}^x |f^{(n)}(u) - f^{(n)}(x)|\left(u - \frac{t}{a_k}\right)^{n-1} du$$

$$\le \int_{t/a_k}^{x} \omega(f^{(n)}; |u-x|) \left(u - \frac{t}{a_k}\right)^{n-1} du$$

$$\le \int_{t/a_k}^{x} \left(u - \frac{t}{a_k}\right)^{n-1} du\, \omega\left(f^{(n)}; \left|\frac{t}{a_k} - x\right|\right) \le \frac{1}{n}\left(x - \frac{t}{a_k}\right)^{n} \omega\left(f^{(n)}; \frac{2\alpha}{a_k}\right).$$

In both cases, taking again the advantage of the first inequality we obtain the desired result. The case $x = t/a_k$ is trivial and the proof is complete. □

We present the main result of this section.

Theorem 3.2. *Let $f \in C^n(\mathbb{R})$. For every $x \in \mathbb{R}$ the operators L_k, $k \in \mathbb{Z}$, defined by (5) verify*

$$|(L_k f)(x) - \gamma^2 \delta f(x)|$$
$$\le \gamma^2 \delta \left(\sum_{i=1}^{n} \frac{|f^{(i)}(x)|}{i!} \left(\frac{2\alpha}{a_k}\right)^{i} + \frac{1}{n!}\left(\frac{2\alpha}{a_k}\right)^{n} \omega\left(f^{(n)}; \frac{2\alpha}{a_k}\right)\right). \tag{11}$$

Proof. Let fix $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Successively based on (5), (3), (4) and (9) we get

$$|(L_k f)(x) - \gamma^2 \delta f(x)| = \Big| \sum_{j\in\mathbb{Z}} (f, \varphi_{k,j})\varphi_{k,j}(x) - \gamma\delta f(x) \sum_{j\in\mathbb{Z}} \varphi(a_k x + j\delta) \Big|$$
$$\le \sqrt{a_k} \sum_{j \in J_{k,x}} |(f, \varphi_{k,j}) - \gamma\delta\sqrt{a_{-k}} f(x)| \varphi(a_k x + j\delta). \tag{12}$$

In the above we also used (1). It is obvious that, in what follows, we are interested only on the indices j belonging to $J_{k,x}$.

With the help of relations (4) and (7) we can write

$$|(f, \varphi_{k,j}) - \gamma\delta\sqrt{a_{-k}} f(x)|$$
$$= \Big| \sqrt{a_{-k}} \int_{\mathbb{R}} f\left(\frac{t}{a_k}\right) \varphi(t + \delta j) dt - \sqrt{a_{-k}} f(x) \int_{\mathbb{R}} \varphi(t+\delta j) dt \Big|$$
$$\le \sqrt{a_{-k}} \int_{\mathbb{R}} \left| f\left(\frac{t}{a_k}\right) - f(x)\right| \varphi(t + \delta j) dt. \tag{13}$$

Choosing in (8) $y := t/a_k$, $t \in [-\alpha - j\delta, \alpha - j\delta]$, and using both (9) and (10) we have

$$\left| f\left(\frac{t}{a_k}\right) - f(x)\right| \le \sum_{i=1}^{n} \frac{|f^{(i)}(x)|}{i!} \left|\frac{t}{a_k} - x\right|^{i} + \frac{1}{(n-1)!} r_n(f; a_k, x, t/a_k)$$
$$\le \sum_{i=1}^{n} \frac{|f^{(i)}(x)|}{i!} \left(\frac{2\alpha}{a_k}\right)^{i} + \frac{1}{n!}\left(\frac{2\alpha}{a_k}\right)^{n} \omega\left(f^{(n)}; \frac{2\alpha}{a_k}\right).$$

Returning on (13) and further on (12), we obtain the claimed result. □

Letting $C_b^n(\mathbb{R}) := \{f \in C^n(\mathbb{R}) | \ f^{(i)} \in B(\mathbb{R}),\ 0 \le i \le n\}$, relation (11) leads us to the following global estimate of the error.

Theorem 3.3. *For every $f \in C_b^n(\mathbb{R})$, the operators L_k, $k \in \mathbb{Z}$, defined by (5) verify*

$$\|L_k f - \gamma^2 \delta f\|_\infty \le \gamma^2 \delta \left(\sum_{i=1}^{n} \frac{\beta_k^i}{i!} \|f^{(i)}\|_\infty + \frac{\beta_k^n}{n!} \omega(f^{(n)}; \beta_k) \right), \tag{14}$$

where $\beta_k := 2\alpha a_{-k}$.

In the above, under the hypothesis $\lim\limits_{k\to\infty} a_k = \infty$, one has $\beta_k < 1$ for sufficiently large k. Considering the semi-norm $|\cdot|_{C_b^n(\mathbb{R})}$ of the vector space $C_b^n(\mathbb{R})$ defined by $|h|_{C_b^n(\mathbb{R})} := \sum\limits_{i=1}^{n} \|h^{(i)}\|_\infty$, relation (14) implies

$$\left\| \frac{1}{\gamma^2 \delta} L_k f - f \right\|_\infty \le \left(\frac{2\alpha}{a_k} \right) \left(|f|_{C_b^n(\mathbb{R})} + \omega \left(f^{(n)}; \frac{2\alpha}{a_k} \right) \right),$$

for every $f \in C_b^n(\mathbb{R})$ and sufficiently large k.

4. **On the degree of exactness**

In what follows, for any integer $s \ge 0$ we denote by e_s the test function defined by $e_s(x) = x^s$, $x \in \mathbb{R}$.

Under an additional assumption, we prove that the operator $(1/\gamma^2\delta)L_k$ reproduces the affine functions, in other words it has the degree of exactness equal to 1. We assume that the scaling function φ of (δ, γ) type has the following property

$$\sum_{j=-\infty}^{\infty} j\varphi(x + \delta j) = -\frac{\gamma}{\delta} x, \quad x \in \mathbb{R}. \tag{15}$$

Lemma 4.1. *Let φ be a scaling function of (δ, γ) type such that condition (15) is fulfilled. One has*

$$\int_{\mathbb{R}} u\varphi(u) du = 0. \tag{16}$$

Proof. We observe that

$$\int_{\mathbb{R}} u\varphi(u)du = \sum_{j\in\mathbb{Z}} \int_{\delta j}^{\delta(j+1)} u\varphi(u)du = \sum_{j\in\mathbb{Z}} \int_0^{\delta} (x+\delta j)\varphi(x+\delta j)dx$$
$$= \int_0^{\delta} x\Big(\sum_{j\in\mathbb{Z}} \varphi(x+\delta j)\Big)dx + \delta \int_0^{\delta} \Big(\sum_{j\in\mathbb{Z}} j\varphi(x+\delta j)\Big)dx.$$

Taking into account identities (3) and (15), the proof is finished. □

We come now to the main result of the section.

Theorem 4.2. *Let L_k, $k \in \mathbb{Z}$, be defined by (5) such that (15) is fulfilled. For every real-valued polynomial p of degree less or equal to 1, one has $L_k p = \gamma^2 \delta p$.*

Proof. Obviously, it is enough to verify the claimed identity only for the monomials e_0 and e_1. For computations we use the formula given at (6).

Based on (7) and (3) one gets $L_k e_0 = \gamma^2 \delta e_0$. The same quoted relations together with (16) guarantee that $L_k e_1 = \gamma^2 \delta e_1$. The conclusion follows. □

At this moment, the idea to present $L_k e_2$ comes out. In order to achieve it, we introduce the function θ given by

$$\theta(x) = \sum_{j=-\infty}^{\infty} j^2 \varphi(x+\delta j), \quad x \in \mathbb{R}. \tag{17}$$

Since (2) takes place, the above sum is finite and θ is well-defined. Moreover, θ is non-negative and belongs to $L_{1,loc}(\mathbb{R})$.

Theorem 4.3. *Let L_k, $k \in \mathbb{Z}$, be defined by (5) such that (15) is fulfilled. If θ is given by (17) then the following identities hold true*

(i) $(L_k e_2)(x) = \dfrac{\gamma}{a_k^2}(\|e_2\varphi\|_{L_1(\mathbb{R})} + \delta^3 \theta(a_k x))$, $x \in \mathbb{R}$, (18)

(ii) $\|e_2\varphi\|_{L_1(\mathbb{R})} = \delta^2 \displaystyle\int_0^{\delta} \theta(t)dt - \frac{\gamma}{3}\delta^3$.

Proof. (i) Clearly, $e_2\varphi \in L_1(\mathbb{R})$. Resorting to (6) we can write

$$(L_k e_2)(x) = \frac{1}{a_k^2} \sum_{j\in\mathbb{Z}} \varphi(a_k x + \delta j)\Big(\|e_2\varphi\|_{L_1(\mathbb{R})} - 2\delta j \int_{\mathbb{R}} u\varphi(u)du + \delta^2 j^2 \|\varphi\|_{L_1(\mathbb{R})}\Big).$$

Taking in view relations (16), (7), (3) and (17) we obtain (18).

(ii) Since $\|e_2\varphi\|_{L_1(\mathbb{R})} = \sum_{j\in\mathbb{Z}} \int_{\delta j}^{\delta(j+1)} u^2\varphi(u)du = \sum_{j\in\mathbb{Z}} \int_0^{\delta} (t+\delta j)^2\varphi(t+\delta j)dt$, with the help of (3), (15) and (17), our statement is proved. $\square$

5. **Estimates for bounded functions**

Based on (6) and (3) we can remark in passing that

$$\|L_k f\|_\infty \le \gamma^2\delta\|f\|_\infty,\ \text{for every } f \in B(\mathbb{R}) \cap L_{1,loc}(\mathbb{R}).$$

Consequently, for $\gamma^2\delta < 1$ each operator L_k is a contraction.

The aim of this section is to give bounds for error approximation by using a Lipschitz-type function introduced by Lenze [5; *Eq. (1.5)*]. We recall this map we will have to deal with. Let $J \subset \mathbb{R}$ be an interval. Let $f \in \mathbb{R}^J$ be bounded and $\mu \in]0,1]$. The Lipschitz-type maximal function of order μ associated to f is defined as

$$f_\mu^\sim(x) := \sup_{\substack{t\neq x\\ t\in J}} \frac{|f(t)-f(x)|}{|t-x|^\mu}, \quad x \in J. \tag{19}$$

The local behaviour of function f can be measured by $f_\mu^\sim$. The finiteness of $f_\mu^\sim$ gives a local control for the smoothness of f. Roughly speaking, the boundedness of $f_\mu^\sim$ is equivalent to $f \in \mathrm{Lip}\mu$ on J.

Theorem 5.1. *Let L_k, $k \in \mathbb{Z}$, be defined by (5) such that (15) is fulfilled. For every $\mu \in]0,1]$ and $f \in B(\mathbb{R}) \cap L_{1,loc}(\mathbb{R})$ the following inequality*

$$|(L_k f)(x) - \gamma^2\delta f(x)| \le M_\varphi \left(\frac{\alpha^2}{3a_k^2} + \frac{\delta^2}{\gamma a_k^2}\theta(a_k x) - x^2\right)^{\mu/2} f_\mu^\sim(x), \quad x \in \mathbb{R},$$

holds true, where $M_\varphi = \gamma(2\alpha)^{\mu/2}\|\varphi\|_{L_p([-\alpha,\alpha])}$, $p = 2/(2-\mu)$ and θ is given at (17).

Proof. Let fix $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. In what follows, for the sake of simplicity, we set

$$(I_{k,j}\varphi)(x) := \int_{\mathbb{R}} \left|\frac{t}{a_k} - x\right|^\mu \varphi(t+\delta j)dt,$$

$$c_{k,j}(x) := \int\limits_{\mathrm{supp}(\varphi)} \left(\frac{t-\delta j}{a_k} - x\right)^2 dt,\ (j \in \mathbb{Z}),$$

and

$$s_k(x) := \sum_{j\in\mathbb{Z}} c_{k,j}(x)\varphi(a_k x + \delta j).$$

In concordance with formula (19) we write

$$|f(t/a_k) - f(x)| \le f_\mu^\sim(x)|t/a_k - x|^\mu.$$

Taking the advantage of relations (12) and (13), one obtains

$$\begin{aligned}|(L_k f)(x) - \gamma^2\delta f(x)| &\le \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} \left| f\left(\frac{t}{a_k}\right) - f(x)\right| \varphi(t+\delta j)dt\varphi(a_k x + \delta j)\\ &\le f_\mu^\sim(x)\sum_{j\in\mathbb{Z}}(I_{k,j}\varphi)(x)\varphi(a_k x+\delta j).\end{aligned}$$

By using Hölder's integral inequality with parameters $q := 2/\mu$ and $p := 2/(2-\mu)$, we deduce

$$(I_{k,j}\varphi)(x)$$

$$= \int\limits_{\mathrm{supp}(\varphi)} \left|\frac{u-\delta j}{a_k} - x\right|^\mu \varphi(u)du \le c_{k,j}^{\mu/2}(x)\Big(\int\limits_{\mathrm{supp}(\varphi)} \varphi^{2/(2-\mu)}(u)du\Big)^{(2-\mu)/2}. \tag{21}$$

The last quantity represents $\|\varphi\|_{L_p([-\alpha,\alpha])}$, see (2).

Further on, based on Hölder's discrete inequality with the same parameters q, p, we have

$$\sum_{j\in\mathbb{Z}} c_{k,j}^{\mu/2}(x)\varphi(a_k x+\delta j) = \sum_{j\in\mathbb{Z}}(c_{k,j}(x)\varphi(a_k x+\delta j))^{\mu/2}\varphi^{1-\mu/2}(a_k x+\delta j)$$

$$\le \Big(\sum_{j\in\mathbb{Z}} c_{k,j}(x)\varphi(a_k x+\delta j)\Big)^{\mu/2}\Big(\sum_{j\in\mathbb{Z}}\varphi(a_k x+\delta j)\Big)^{(2-\mu)/2} = \gamma^{(2-\mu)/2}s_k^{\mu/2}(x). \tag{22}$$

In order to evaluate $s_k(x)$, we shall use (3), (15), (18) and (2).

$$s_k(x) = \frac{1}{a_k^2}\int\limits_{\mathrm{supp}(\varphi)}\sum_{j\in\mathbb{Z}}(t-\delta j - a_k x)^2\varphi(a_k x+\delta j)dt$$

$$= \frac{1}{a_k^2}\int\limits_{\mathrm{supp}(\varphi)}(t^2\gamma + \delta^2\theta(a_k x) - a_k^2\gamma x^2)dt \le \frac{2\alpha\gamma}{a_k^2}\left(\frac{\alpha^2}{3} + \frac{\delta^2}{\gamma}\theta(a_k x) - a_k^2 x^2\right). \tag{23}$$

Collecting (23), (22), (21) and substituting in (20) we finish the proof. □

Remarks. (i) In the particular case $\gamma = \delta = 1$, $a_k = 2^k$, L_k turns into Anastassiou's original operator A_k, [2; §*6.1*]. As far as we know, Theorem 5.1 establishes a new result for A_k which involves Lenze's function.

(ii) For comparison, it is a standard fact in Littlewood-Paley theory that if f and φ are both Hölder continuous of order $\mu > 0$ and if φ has compact support then the μ-Hölder norm of $L_k f - f$ decays like $1/a_k$.

(iii) Regarding this note, we mention that a similar approach could be made by considering the following $(L_a)_{a>0}$ net of operators, $L_a f := \sum_{j\in\mathbb{Z}} (f, \varphi_{j,a})\varphi_{j,a}$ with $\varphi_{j,a}(x) = \sqrt{a}\varphi(ax + j\delta)$. The estimates would be exactly the same.

References

[1] Agratini, O., Operators generated by a quasi-scaling type function, *Revista de la Unión Matemática Argentina*, **44**(2003), 2, 21-30.

[2] Anastassiou, G., *Quantitative Approximations*, Chapman & Hall/CRC, Boca Raton, 2001.

[3] Anastassiou, G., Yu, X. M., Monotone and probabilistic wavelet approximation, *Stochastic Anal. Appl.*, **10**(1992), 251-264.

[4] Daubechies, I., *Ten Lectures on Wavelets*, NSF-CBMS Regional Conference Series in Applied Math. **61**, SIAM Publications, Philadelphia, 1992.

[5] Lenze, B., Bernstein-Baskakov-Kantorovič operators and Lipschitz-type maximal functions, *Colloquia Mathematica Societatis János Bolyai 58, Approximation Theory*, Kecskemèt, Hungary (J. Szabados, K. Tandori, eds.), pp. 469-496, North-Holland Publishing Company, Amsterdam – Oxford – New York, 1990.

Babeş-Bolyai University, Faculty of Mathematics
and Computer Science, Str. Kogălniceanu, 1
Cluj-Napoca, Romania
E-mail address: agratini@math.ubbcluj.ro