STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

ON GENERALIZED DIFFERENCE LACUNARY STATISTICAL CONVERGENCE

BINOD CHANDRA TRIPATHY AND MIKÂIL ET

Abstract. A lacunary sequence is an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$, $k_r - k_{r-1} \to \infty$ as $r \to \infty$. A sequence x is called $S_\theta(\Delta^m)-$ convergent to L provided that for each $\varepsilon > 0$, $\lim_r (k_r - k_{r-1})^{-1}$ {the number of $k_{r-1} < k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\} = 0$, where $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$. The purpose of this paper is to introduce the concept of Δ^m- lacunary statistical convergence and Δ^m-lacunary strongly convergence and examine some properties of these sequence spaces. We establish some connections between Δ^m-lacunary strongly convergence and Δ^m-lacunary statistical convergence. It is shown that if a sequence is Δ^m-lacunary strongly convergent then it is Δ^m-lacunary statistically convergent. We also show that the space $S_\theta(\Delta^m)$ may be represented as a $[f, p, \theta](\Delta^m)$ space.

1. Introduction

Throughout the article w, ℓ_∞, c, c_0, $\bar{c}$, and $\bar{c}_0$ denote the spaces of all, bounded, convergent, null, statistically convergent and statistically null complex sequences. The notion of statistical convergence was introduced by Fast [6] and Schoenberg [19] independently. Subsequently statistical convergence have been discussed in ([5], [7], [8], [12], [16], [18]).

Received by the editors: 15.11.2004.

2000 *Mathematics Subject Classification.* 40A05, 40C05, 46A45.

Key words and phrases. Difference sequence, statistical convergence, lacunary sequence.

The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. A subset E of $\mathbb{N}$ is said to have density $\delta(E)$, if

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k) \text{ exists,}$$

where χ_E is the characteristic function of E.

A sequence (x_n) is said to be statistically convergent to L if for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$. In this case we write $S - \lim x_k = L$ or $x_k \to L(S)$.

The notion of difference sequence spaces was introduced by Kizmaz [10]. Later on the notion was generalized by Et and Çolak [3] and was studied by Et and Basarir [4], Malkowsky and Parashar [14], Et and Nuray [5], Çolak [2] and many others.

Let m be a non-negative integer, then

$$X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

for $X = \ell_\infty$, c and c_0, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$ and $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$.

The generalized difference has the following binomial representation:

$$\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}.$$

The sequence spaces $\ell_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ are BK-spaces, normed by

$$\|x\|_\Delta = \sum_{i=0}^{m} |x_i| + \|\Delta^m x\|_\infty .$$

We call these sequence spaces Δ^m-bounded, Δ^m-convergent and Δ^m-null sequences, respectively. The classes $\bar{c}(\Delta^m)$ and $\bar{c}_0(\Delta^m)$ was studied by Et and Nuray [5].

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio k_r/k_{r-1} will be denoted by q_r.

Let $E, F \subset w$. Then we shall write

$$M(E,F) = \underset{x\in E}{\cap} x^{-1} * F = \{a \in w : ax \in F \text{ for all } x \in E\} \text{ [20]}.$$

The set $E^{\alpha} = M(E, l_1)$ is called Köthe-Toeplitz dual space or $\alpha-$dual of E.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$,

A sequence space E is said to be symmetric if $(x_k) \in E$ implies $\left(x_{\pi(k)}\right) \in E$, where $\pi(k)$ is a permutation of $\mathbb{N}$,

A sequence space E is said to be convergence free when, if x is in E and if $y_k = 0$ whenever $x_k = 0$, then y is in E,

A sequence space E is said to be monotone if it contains the canonical preimages of its step spaces,

A sequence space E is said to be sequence algebra if $x.y \notin E$ whenever $x, y \in E$,

A sequence space E is said to be perfect if $E = E^{\alpha\alpha}$ [9].

It is well known that if E is perfect $\Longrightarrow E$ is normal.

The following inequality will be used throughout this paper.

$$|a_k + b_k|^{p_k} \leq C\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \tag{1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p_k = H$, $C = \max\left(1, 2^{H-1}\right)$.

The notion of modulus function was introduced by Nakano [15]. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such taht

i) $f(x) = 0$ if and only if $x = 0$, ii) $f(x+y) \leq f(x) + f(y)$ for $x, y \geq 0$, iii) f is increasing, iv) f is continuous from the right at 0.

It follows that f must be continuous everwhwre on $[0,\infty)$. A modulus may be unbounded or bounded. Ruckle [17] and Maddox [12] used a modulus f to construct sequence spaces.

2. **Definitions and Preliminaries**

The notion of almost convergence of sequences was introduced by Lorentz [11]. The notion was generalized by Et and Başarır [4].

Definition 2.1 [4] The sequence (x_n) is said to be Δ^m-almost convergent to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=k+1}^{k+n}(\Delta^m x_i - L) = 0, \text{ uniformly in } k.$$

We denote the class of all Δ^m-almost convergent sequences by $AC(\Delta^m)$.

Definition 2.2 [4] The sequence (x_n) is said to be Δ^m- strongly almost convergent to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=k+1}^{k+n}|\Delta^m x_i - L| = 0, \text{ uniformly in } k.$$

We denote the class of all Δ^m-strongly almost convergent sequences by $|AC|(\Delta^m)$.

Definition 2.3 [8] The sequence (x_k) is said to be lacunary statistically convergent to L if for each $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\operatorname{card}\{k \in I_r : |x_k - L| \geq \varepsilon\} = 0.$$

The class of all lacunary statistically convergent sequences is denoted by S_θ.

Definition 2.4 A sequence (x_n) is said to be Δ^m-Cesàro summable to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(\Delta^m x_k - L) = 0.$$

The class of all Δ^m-Cesàro summable sequences is denoted by $\sigma_1(\Delta^m)$.

Definition 2.5 A sequence (x_n) is said to be Δ^m-strongly Cesàro summable to L if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\Delta^m x_k - L| = 0.$$

The class of all Δ^m-strongly Cesàro summable sequences is denoted by $|\sigma_1|(\Delta^m)$.

Now we introduce the definitions of Δ^m-lacunary statistically convergence, Δ^m- lacunary strongly convergence and Δ^m-lacunary strongly convergence with respect to a modulus f.

Definition 2.6 Let θ be a lacunary sequence, the number sequence x is Δ^m-lacunary statistically convergent to the number L provided that for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \operatorname{card}\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\} = 0.$$

In this case we write $S_\theta(\Delta^m) - \lim x_k = L$ or $x_k \to L(S_\theta(\Delta^m))$. We denote Δ^m-lacunary statistically convergent sequence by $S_\theta(\Delta^m)$.

Definition 2.7 Let θ be a lacunary sequence. Then a sequence (x_k) is said to be $C_\theta(\Delta^m)-$summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} (\Delta^m x_k - L) = 0.$$

We denote the class of all $C_\theta(\Delta^m)-$summable sequences by $C_\theta(\Delta^m)$.

A sequence (x_k) is said to be Δ^m- lacunary strongly summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\Delta^m x_k - L| = 0.$$

We denote the class of all Δ^m- lacunary strongly summable sequences by $N_\theta(\Delta^m)$. In the case $L = 0$ we shall write $N_\theta^0(\Delta^m)$ instead of $N_\theta(\Delta^m)$. It can be shown that the sequence space $N_\theta(\Delta^m)$ is a Banach space with norm by

$$\|x\|_{\Delta\theta} = \sum_{i=1}^{m} |x_i| + \sup_r \frac{1}{h_r} \sum_{k\in I_r} |\Delta^m x_k|.$$

If we take $m = 0$ then we obtain the sequence space N_θ which were introduced by Freedman et al.[1].

Definition 2.8 Let f be a modulus function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence set

$$[f, p, \theta](\Delta^m) = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} [f(|\Delta^m x_k - L|)]^{p_k} = 0, \quad \text{for some } L \right\},$$

If $x \in [f, p, \theta](\Delta^m)$, then we will write $x_k \to L[f, p, \theta](\Delta^m)$ and will be called Δ^m-lacunary strongly summable with respect to a modulus f. In the case $p_k = 1$ for all $k \in \mathbb{N}$, we shall write $[f, \theta](\Delta^m)$ instead of $[f, p, \theta](\Delta^m)$. It may be noted here that the space $[f, \theta](\Delta^m)$ was discussed by Colak [2].

3. **Main Results**

In this section we prove the results of this article. The proof of the following results is a routine work.

Proposition 3.1 Let θ be a lacunary sequence, then $S_\theta\left(\Delta^{m-1}\right) \subset S_\theta\left(\Delta^m\right)$. In general $S_\theta\left(\Delta^i\right) \subset S_\theta\left(\Delta^m\right)$, for all $i = 1, 2, \ldots, m-1$. Hence $S_\theta \subset S_\theta\left(\Delta^m\right)$ and the inclusions are strict.

Theorem 3.2 If a Δ^m-bounded sequence is Δ^m-statistically convergent to L then it is Δ^m-Cesàro summable to L.

Proof. Without loss of generality we may assume that $L = 0$. Then,

$$\begin{aligned} \left|\frac{1}{n}\sum_{k=1}^{n}\Delta^m x_k\right| &\leq \frac{1}{n}\sum_{k=1}^{n}\left|\Delta^m x_k\right| = \frac{1}{n}\sum_{\substack{1\leq k\leq n\\ |\Delta^m x_k|\geq\varepsilon}}\left|\Delta^m x_k\right| + \frac{1}{n}\sum_{\substack{1\leq k\leq n\\ |\Delta^m x_k|<\varepsilon}}\left|\Delta^m x_k\right| \\ &< \frac{1}{n}K\,\mathrm{card}\left\{k\leq n : \left|\Delta^m x_k\right|\geq\varepsilon\right\} + \frac{n}{n}\varepsilon. \end{aligned}$$

Thus $x \in \sigma_1\left(\Delta^m\right)$. Converse of Theorem 3.2 does not holds, for example, the sequence $x = (0, -1, -1, -2, -2, -3, -3, -4, -4, ...)$ belongs to $\sigma_1(\Delta)$ and does not belong to $S(\Delta)$.

Theorem 3.3 Let θ be a lacunary sequence, then

i) If a sequence is Δ^m-lacunary strongly convergent to L, then it is Δ^m-lacunary statistically convergent to L and the inclusion is strict.

ii) If a Δ^m-bounded sequence is Δ^m-lacunary statistically convergent to L then it is Δ^m-lacunary strongly convergent to L.

iii) $\ell_\infty\left(\Delta^m\right) \cap S_\theta\left(\Delta^m\right) = \ell_\infty\left(\Delta^m\right) \cap N_\theta\left(\Delta^m\right)$.

Proof. We give the proof of (i) only. If $\varepsilon > 0$ and $x_k \to L\left(N_\theta\left(\Delta^m\right)\right)$ we can write

$$\sum_{k\in I_r}\left|\Delta^m x_k - L\right| \geq \sum_{\substack{k\in I_r\\ |\Delta^m x_k - L|\geq\varepsilon}}\left|\Delta^m x_k - L\right| \geq \varepsilon.\left|\left\{k\in I_r : \left|\Delta^m x_k - L\right|\geq\varepsilon\right\}\right|.$$

Hence $x_k \to L\left(S_\theta\left(\Delta^m\right)\right)$. The inclusion is strict. In order to establish this, let θ be given and define $\Delta^m x_k$ to be $1, 2, ..., \left[\sqrt{h_r}\right]$ at the first $\left[\sqrt{h_r}\right]$ integers in I_r, and $\Delta^m x_k = 0$ otherwise. Then x is not Δ^m-bounded, $x_k \to 0\,(S_\theta\,(\Delta^m))$ and $x_k \nrightarrow 0\,(N_\theta\,(\Delta^m))$.

Note that any Δ^m-bounded $S_\theta\,(\Delta^m)-$summable sequence is $C_\theta\,(\Delta^m)-$summable.

Theorem 3.4 Let θ be a lacunary sequence, then $S\,(\Delta^m) = S_\theta\,(\Delta^m)$ if and only if $1 < \lim_r \inf q_r \leq \lim_r \sup q_r < \infty$.

The proof of Theorem 3.4, we need the following lemmas.

Lemma 3.5 For any lacunary sequence θ, $S\,(\Delta^m) \subset S_\theta\,(\Delta^m)$ if and only if $\lim_r \inf q_r > 1$.

Proof. If $\liminf_r q_r > 1$ there exists a $\delta > 0$ such that $1 + \delta \leq q_r$ for sufficiently large r. Since $h_r = k_r - k_{r-1}$, we have $\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta}$. Let $x_k \to L\,(S_\theta\,(\Delta^m))$. Then for every $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{k_r}\left|\{k \leq k_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| &\geq \frac{1}{k_r}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\{k \in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right|.\end{aligned}$$

Hence $S\,(\Delta^m) \subset S_\theta\,(\Delta^m)$.

Conversely suppose that $\liminf_r q_r = 1$. If we consider the sequence defined by,

$$\Delta^m x_i = \begin{cases} 1, & \text{if } i \in I_{r_j} \quad \text{for some } j = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases}$$

then $x \in \ell_\infty\,(\Delta^m)$ but $x \notin N_\theta\,(\Delta^m)$. However, $x \in |\sigma_1|\,(\Delta^m)$. Theorem 3.3 (ii) implies that $x \notin S_\theta\,(\Delta^m)$. On the other hand if a sequence is strongly Δ^m-strongly Cesàro summable to L then it is Δ^m-statistically convergent to L (Theorem 4.2, Et and Nuray [5]). Hence $S\,(\Delta^m) \nsubseteq S_\theta\,(\Delta^m)$ and the proof is complete.

Lemma 3.6 For any lacunary sequence θ, $S_\theta\,(\Delta^m) \subset S\,(\Delta^m)$ if and only if $\limsup_r q_r < \infty$.

Proof. Sufficiency can be proved using the same technique of Lemma 3 of [8]. Now suppose that $\limsup_r q_r = \infty$. Consider the sequence defined by

$$\Delta^m x_i = \begin{cases} 1, & \text{if } k_{r_j-1} < i \leq 2k_{r_j-1} \quad \text{for some } j = 1,2,3,... \\ 0, & \text{otherwise} \end{cases}.$$

Then $x \in N_\theta(\Delta^m)$ but $x \notin |\sigma_1|(\Delta^m)$. Clearly we have $x \in S_\theta(\Delta^m)$, but Theorem 4.2 of Et and Nuray [5] $x \notin S(\Delta^m)$. Hence $S_\theta(\Delta^m) \nsubseteq S(\Delta^m)$. This completes the proof.

Lemma 3.7 If £ denotes the set of all lacunary sequences, then

$$|AC|(\Delta^m) = \ell_\infty(\Delta^m) \cap (\cap_{\theta \in £} S_\theta(\Delta^m)).$$

Proof. Omitted.

Lemma 3.8 Let E be any of the spaces σ_1, $|\sigma_1|$, C_θ, N_θ, N_θ^0, AC, $|AC|$ and S_θ. Then the sequence spaces $E(\Delta^m)$ are neither solid nor symmetric nor sequence algebra nor convergence free nor perfect.

Proof. Proof follows from the following examples.

Example 1. Let $\theta = (2^r)$. Then $x = (k) \in N_\theta^0(\Delta^2)$, but $\alpha x = (\alpha_k x_k) \notin N_\theta^0(\Delta^2)$, for $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$. Hence $N_\theta^0(\Delta^m)$ is not solid.

Example 2. Let $\theta = (2^r)$. Then $x = (k) \in (N_\theta)(\Delta)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows:

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, ...\}.$$

Then $(y_k) \notin (N_\theta)(\Delta)$.

Example 3. Let $\theta = (2^r)$. Then $x = (k) \in N_\theta^0(\Delta^2)$. Let (y_k) be a rearrangement of (x_k), which is defined as above, then $(y_k) \notin N_\theta^0(\Delta^2)$.

Example 4. Let $\theta = (2^r)$. Consider the sequences $x = (k)$, $y = (k^{m-1})$, then $x, y \in N_\theta^0(\Delta^m)$ but $x.y \notin N_\theta^0(\Delta^m)$. For the others spaces consider the sequences $x = (k)$, $y = (k^m)$.

Example 5. Let $\theta = (2^r)$. Then $(x_k) = (1)$ is in $N_\theta^0(\Delta)$. The sequence (y_k) defined as $y_k = k$ for all $k \in \mathbb{N}$ does not belong to $N_\theta^0(\Delta)$. Hence $N_\theta^0(\Delta)$ is not convergence free.

Note. Similarly different examples can be constructed for the other spaces.

Now we will give some relations between Δ^m-lacunary statistically convergent sequences and Δ^m- lacunary strongly summable sequences with respect to a modulus function.

Theorem 3.9 The inclusion $[f,p,\theta]\left(\Delta^{m-1}\right) \subset [f,p,\theta]\left(\Delta^m\right)$ is strict. In general $[f,p,\theta]\left(\Delta^{i}\right) \subset [f,p,\theta]\left(\Delta^m\right)$ for all $i = 1,2,\ldots,m-1$ and the inclusion is strict.

Proof. Straight forward and hence omitted.

Theorem 3.10 Let f, f_1, f_2 be modulus functions. Then we have

i) $[f,\theta]\left(\Delta^m\right) \subset [f\circ f_1,\theta]\left(\Delta^m\right)$,

ii) $[f_1,p,\theta]\left(\Delta^m\right) \cap [f_2,p,\theta]\left(\Delta^m\right) \subset [f_1+f_2,p,\theta]\left(\Delta^m\right)$.

Proof. i) Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \le t \le \delta$. Write $y_k = f_1\left(\left|\Delta^m x_k - L\right|\right)$ and consider

$$\sum_{k\in I_r} f(y_k) = \sum_1 f(y_k) + \sum_2 f(y_k)$$

where the first summation is over $y_k \le \delta$ and second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 f(y_k) < h_r\varepsilon \tag{2}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \le 1 + \frac{y_k}{\delta}.$$

By the definition of f we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$\sum_2 f(y_k) \le 2f(1)\delta^{-1}\sum_{k=1}^{n} y_k. \tag{3}$$

From(2) and (3), we obtain $[f,\theta]\left(\Delta^m\right) \subset [f\circ f_1,\theta]\left(\Delta^m\right)$.

ii) The proof of (ii) follows from the following inequality

$$\left[\left(f_1+f_2\right)\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k} \le C\left[f_1\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k} + C\left[f_2\left(\left|\Delta^m x_k - L\right|\right)\right]^{p_k}.$$

The following result is a consequence of Theorem 3.10 (i).

Proposition 3.11 ([2]) Let f be a modulus function. Then $N_\theta(\Delta^m) \subset [f,\theta](\Delta^m)$.

Theorem 3.12 Let $0 < p_k \leq q_k$ and (q_k/p_k) be bounded. Then $[f,q,\theta](\Delta^m) \subset [f,p,\theta](\Delta^m)$.

Proof: If we take $w_k = [f(|\Delta^m x_k - L|)]^{q_k}$ for all k. Following the technique applied for establishing Theorem 5 of Maddox [13], we can easily prove the theorem.

Theorem 3.13 The sequence space $[f,p,\theta](\Delta^m)$ is neither solid nor symmetric nor sequence algebra nor convergence free nor perfect for $m \geq 1$.

To show these, consider the examples cited in Lemma 3.8.

Theorem 3.14 Let f be modulus function and $\sup_k p_k = H$. Then $[f,p,\theta](\Delta^m) \subset S_\theta(\Delta^m)$.

Proof. Let $x \in [f,p,\theta](\Delta^m)$ and $\varepsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r}[f(|\Delta^m x_k - L|)]^{p_k} = \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |\Delta^m x_k - L|\geq\varepsilon}}[f(|\Delta^m x_k - L|)]^{p_k}$$

$$+\frac{1}{h_r}\sum_{k\in I_r |\Delta^m x_k - L| < \varepsilon}[f(|\Delta^m x_k - L|)]^{p_k}$$

$$\geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |\Delta^m x_k - L|\geq\varepsilon}}[f(|\Delta^m x_k - L|)]^{p_k} \geq \frac{1}{h_r}\sum_{k\in I_r}[f(\varepsilon)]^{p_k}$$

$$\geq \frac{1}{h_r}\sum_{k\in Ir}\min\left([f(\varepsilon)]^{\inf p_k}, [f(\varepsilon)]^H\right)$$

$$\geq \frac{1}{h_r}\left|\{k\in I_r : |\Delta^m x_k - L| \geq \varepsilon\}\right| \min\left([f(\varepsilon)]^{\inf p_k}, [f(\varepsilon)]^H\right).$$

Hence $x \in S_\theta(\Delta^m)$.

Theorem 3.15 Let f be bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_\theta(\Delta^m) \subset [f,p,\theta](\Delta^m)$.

Proof. Suppose that f is bounded and let $\varepsilon > 0$ be given. Then

$$\frac{1}{h_r}\sum_{k\in I_r}[f(|\Delta^m x_k - L|)]^{p_k} = \frac{1}{h_r}\sum_{k\in I_r |\Delta^m x_k - L| \geq \varepsilon}[f(|\Delta^m x_k - L|)]^{p_k}$$

$$+\frac{1}{h_r}\sum_{k\in I_r|\Delta^m x_k-L|<\varepsilon}[f(|\Delta^m x_k-L|)]^{p_k}$$

$$\leq\frac{1}{h_r}\sum_{k\in I_r}\max\left(K^h,K^H\right)+\frac{1}{h_r}\sum_{k\in I_r}[f(\varepsilon)]^{p_k}$$

$$\leq\max\left(K^h,K^H\right)\frac{1}{h_r}\left|\{k\in I_r:|\Delta^m x_k-L|\geq\varepsilon\}\right|$$

$$+\max\left(f(\varepsilon)^h,\ f(\varepsilon)^H\right).$$

Hence $x\in[f,p,\theta](\Delta^m)$.

Theorem 3.16 Let f be bounded and $0<h=\inf_k p_k\leq p_k\leq\sup_k p_k=H<\infty$. Then $S_\theta(\Delta^m)=[f,p,\theta](\Delta^m)$ if and only if f is bounded.

Proof. Let f be bounded. By Theorem 3.14 and Theorem 3.15 we have $S_\theta(\Delta^m)=[f,p,\theta](\Delta^m)$.

Conversely suppose that f is unbounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k)=k^2$, for $k=1,2,\ldots$. If we choose

$$\Delta^m x_i=\begin{cases}t_k, & i=k^2,\ i=1,2,\ldots\\ 0, & \text{otherwise}\end{cases}$$

then we have

$$\frac{1}{n}\left|\{k\leq n:|\Delta^m x_k|\geq\varepsilon\}\right|\leq\frac{\sqrt{n}}{n}$$

for all n and so $x\in S_\theta(\Delta^m)$, but $x\notin[f,p,\theta](\Delta^m)$ for $\theta=(2^r)$ and $p_k=1$ for all $k\in\mathbb{N}$. This contradicts to $S_\theta(\Delta^m)=[f,p,\theta](\Delta^m)$.

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Mathematical Sciences Division, Institute of Advanced Study in Science and Technology, Khanpara, Guwahati 781022 India
E-mail address: `tripathybc@yahoo.com`

Department of Mathematics, Firat University, 23119 Elaziğ Turkey
E-mail address: `mikailet@yahoo.com`