STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

AN APPLICATION OF BRIOT-BOUQUET DIFFERENTIAL SUPERORDINATIONS AND SANDWICH THEOREM

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Abstract. Let $f \in A$. We consider the following integral operator

$$F(z) = \frac{2}{z}\int_0^z f(t)dt. \tag{1}$$

By using this integral operator we obtain a Briot-Bouquet differential superordination and sandwich theorem.

1. **Introduction**

Let $\mathcal{H}(U)$ denote the class of functions analytic in the unit disc

$$U = \{z \in \mathbb{C}, |z| < 1\}.$$

For n a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in H(U):\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots,\ z \in U\},$$

and $A_n = \{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ with $A_1 = A$.

A function $f \in \mathcal{H}[a, n]$ is convex in U if it is univalent and $f(U)$ is convex. It is well known that f is convex if and only if $f'(0) \neq 0$ and

$$\mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0, \quad z \in U.$$

Let Q denote the set of functions f that are analytic and injective on the set $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U,\ \lim_{z \to \zeta} f(z) = \infty\right\}$$

Received by the editors: 03.12.2004.

2000 *Mathematics Subject Classification.* Primary 30C80, Secondary 30C45, 30A20, 34A40.

Key words and phrases. Differential subordination, differential superordination, Briot-Bouquet, univalent, convex.

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$. The subclass of Q for which $f(0)=a$ is denoted by $Q(a)$.

Many of the inclusion results that follow can be written very easily in terms of subordination and superordination. We recall these definitions. Let $f,F\in\mathcal{H}(U)$ and let F be univalent in U. The function F is said to be superordinate to f, or f is subordinate to F, written $f\prec F$, if $f(0)=F(0)$ and $f(U)\subset F(U)$.

Let β and γ be complex numbers. Let Ω_2 and Δ_2 be sets in the complex plane, and let p be analytic in the unit disc U. In a series of articles the authors and many others [1, p. 80-119] have determined conditions so that

$$\left\{p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\mid z\in U\right\}\subset\Omega_2\ \Rightarrow\ p(U)\subset\Delta_2. \tag{2}$$

The differential operator on the left is known as the Briot-Bouquet differential operator. The main concern in this subject is to find the smallest set Δ_2 in $\mathbb{C}$ for which (2) holds.

In [2] the authors consider the dual problem of determining conditions so that

$$\Omega_1\subset\left\{p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\mid z\in U\right\}\ \Rightarrow\ \Delta_1\subset p(U). \tag{3}$$

In particular we are interested in determining the largest set Δ_1 in $\mathbb{C}$ for which (3) holds.

If the sets Ω and Δ in (2) and (3) are simply connected domains not equal to $\mathbb{C}$, then it is possible to rephrase these expressions very neatly in terms of subordination and superordination in the forms:

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h_2(z)\ \Rightarrow\ p(z)\prec q_2(z) \tag{2'}$$

$$h(z)\prec p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\ \Rightarrow\ q_1(z)\prec p(z). \tag{3'}$$

The left side of $(2')$ is called a Briot-Bouquet differential subordination, and the function q_2 is called a dominant of the differential subordination. The best dominant which provides a sharp result, is the dominant that is subordinate to all other dominants.

In a recent paper [3] the authors introduced the dual concept of a differential superordination. In light of those results we call the left side of (3′) a Briot-Bouquet differential superordination, and the function q, is called a subordinant of the differential superordination. The best subordinant, which provides a sharp result is the subordinant which is superordinate to all other subordinants.

In [3] the authors combine (2′) and (3′) and obtain a condition so that the Briot-Bouquet sandwich

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \tag{4}$$

implies that $q_1(z) \prec p(z) \prec q_2(z)$.

In order to prove the new results we shall use the following lemma:

Lemma A. [3, Corollary 1.1] *Let $\beta, \gamma \in \mathbb{C}$ and let h be convex in U, with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z)$$

has a univalent solution q that satisfies $q(0) = a$ and $q(z) \prec h(z)$. If $p \in \mathcal{H}[a, 1] \cap Q$ and $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U, then

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma}$$

implies

$$q(z) \prec p(z).$$

The function q is the best subordinant.

Lemma B. [1, Th. 3.2.b, p. 83] *Let h be a convex function in U, with $h(0) = a$ and let n be a positive integer. Suppose that the Briot-Bouquet differential equation*

$$q(z) + \frac{nzq'(z)}{q(z) + 1} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$.

If $p \in \mathcal{H}[a, n]$ satisfies

$$p(z) + \frac{zp'(z)}{p(z) + 1} \prec h(z)$$

then $p(z) \prec q(z)$ *and* q *is the best* (a, n) *dominant.*

2. **Main results**

Theorem 1. *Let* $R \in (0, 1]$ *and let* h *be convex in* U*, with* $h(0) = 1$*, defined by*

$$h(z) = 1 + Rz + \frac{zR}{2 + Rz}, \quad z \in U.$$

If $f \in A$ *and* $\frac{zf'(z)}{f(z)}$ *is univalent,* $\frac{zF'(z)}{F(z)} \in \mathcal{H}[1, 1] \cap Q$ *and*

$$h(z) \prec \frac{zf'(z)}{f(z)}, \quad z \in U \tag{5}$$

then

$$q(z) = 1 + Rz \prec \frac{zF'(z)}{F(z)}, \quad z \in U,$$

where F *is given by (1).*

The function q *is the best subordinant.*

Proof. In [4] the authors have showed that

$$h(z) = 1 + Rz + \frac{zR}{2 + Rz}, \quad R \in (0, 1] \tag{6}$$

is convex, and $q(z) = 1 + Rz$ is a univalent solution of (3) which satisfies $q(0) = 1$ and $q(z) \prec h(z)$, $z \in U$.

From (1) we have

$$zF(z) = 2 \int_0^z f(t)dt, \quad z \in U.$$

By using the derivative of this equality, with respect to z, after a short calculation, we obtain

$$zF'(z) + F(z) = 2f(z).$$

This equality is equivalent to

$$F(z) \left[1 + \frac{zF'(z)}{F(z)}\right] = 2f(z), \quad z \in U. \tag{7}$$

If we let

$$p(z) = \frac{zF'(z)}{F(z)}, \tag{8}$$

then (7) becomes

$$F(z)[1+p(z)] = 2f(z), \quad z \in U. \tag{9}$$

By using the derivative of (9) with respect to z, after a short calculation, we obtain

$$\frac{zF'(z)}{F(z)} + \frac{zp'(z)}{1+p(z)} = \frac{zf'(z)}{f(z)}$$

which, using (8), is equivalent to

$$p(z) + \frac{zp'(z)}{1+p(z)} = \frac{zf'(z)}{f(z)}.$$

Using (5) we have

$$1 + Rz + \frac{Rz}{2+Rz} \prec p(z) + \frac{zp'(z)}{1+p(z)}, \quad z \in U.$$

By using Lemma A we deduce that

$$q(z) \prec p(z) = \frac{zF'(z)}{F(z)}, \quad 1 + Rz \prec \frac{zF'(z)}{F(z)}.$$

Theorem 2. *Let h be convex in U, with $h(0) = 1$, defined by*

$$h(z) = 1 + z + \frac{z}{z+2}, \quad z \in U.$$

If $f \in A$ and

$$\frac{zf'(z)}{f(z)} \prec h(z), \quad z \in U \tag{10}$$

then

$$\frac{zF'(z)}{F(z)} \prec 1 + z,$$

where F is given by (1). The function $q(z) = 1 + z$ is best dominant.

Proof. In [4] the authors have showed that

$$h(z) = 1 + z + \frac{z}{z+2}$$

is convex.

From (1) we have

$$zF(z) = 2\int_0^z f(t)dt, \quad z \in U.$$

Following the steps from the proof of Theorem 1 we obtain:

$$p(z)+\frac{zp'(z)}{1+p(z)}=\frac{zf'(z)}{f(z)}.$$

Using (10) we have

$$p(z)+\frac{zp'(z)}{1+p(z)}\prec h(z).$$

By applying Lemma B we obtain

$$p(z)=\frac{zF'(z)}{F(z)}\prec q(z)=1+z,\quad z\in U.$$

The function $q(z)=1+z$ is the best dominant.

Using the conditions from Theorem 1 and Theorem 2 we can write the following

Corollary. *If $f\in A$ and*

$$1+Rz+\frac{zR}{2+Rz}\prec\frac{zf'(z)}{f(z)}\prec 1+z+\frac{z}{2+z},\quad z\in U$$

then

$$1+Rz\prec\frac{zF'(z)}{F(z)}\prec 1+z,\quad z\in U.$$

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