STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

ON THE INVARIANCE PROPERTY OF THE FISHER INFORMATION (II)

CRISTINA-IOANA FĂTU

Abstract. The objective of this paper is to give some properties for the Fisher's information measure when $X_{a\leftrightarrow b}$ represents a bilateral truncated random variable that corresponds to a normal random variable X with the probability density function $f(x;\theta)$, where $\theta = (m, \sigma^2)$, $\theta \in D_\theta$, $D_\theta \subseteq \mathbb{R}^2$, $m \in \mathbb{R}, m-$known parameter, $\sigma^2 \in \mathbb{R}^+, \sigma^2-$unknown parameter.

1. Bilateral truncation effect of a normal distribution on Fisher's information

The Fisher's invariance property will be studied in the case of a truncated normal distribution.

Let X be a normal distribution with probability density function

$$f(x;m,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}, x \in \mathbb{R}, \tag{1}$$

where the parameters m and σ have their usual significance, namely: $m = E(X)$, $\sigma^2 = Var(X)$, $m \in \mathbb{R}$, $\sigma > 0$.

Definition 1. [2] *We say that the random variable X has a normal distribution truncated to the left at $X = a, a \in \mathbb{R}$ and to the right at $X = b, b \in \mathbb{R}$, denoted by $X_{a\leftrightarrow b}$, if its probability density function, denoted by $f_{a\leftrightarrow b}(x;m,\sigma^2)$, has the form*

$$f_{a\leftrightarrow b}(x;m,\sigma^2) = \begin{cases} \dfrac{k(a,b)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} & if\ a \le x \le b, \\ 0 & if\ x < a\ or\ x > b, \end{cases} \tag{2}$$

Received by the editors: 14.03.2005.
2000 *Mathematics Subject Classification.* 62B10, 62B05.
Key words and phrases. Fisher's information, truncated distribution, invariance property.

where

$$k(a,b) = \frac{1}{A} = \frac{1}{\Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right)}, \tag{3}$$

and

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{t^2}{2}\right) dt, \tag{4}$$

is the standard normal distribution function.

Theorem 1. *If the random variable $X_{a\leftrightarrow b}$ has a bilateral truncated normal distribution, that is its probability distribution has the form (2), then the Fisher's information measure about the unknown parameter σ^2,then the parameter m is known, has the following form*

$$\begin{aligned} I_{X_{a\leftrightarrow b}}(\sigma^2) = &-\frac{(a-m)^3 f(a;m,\sigma^2) - (b-m)^3 f(b;m,\sigma^2)}{4\sigma^6 A} - \\ &- \frac{3}{4\sigma^4}\left\{\frac{[(b-m)f(b;m,\sigma^2) - (a-m)f(a;m,\sigma^2)] - A}{A}\right\} - \\ &- \frac{1}{4\sigma^4}\left\{\frac{[(b-m)f(b;m,\sigma^2) - (a-m)f(a;m,\sigma^2)] - A}{A}\right\}^2, \end{aligned} \tag{5}$$

where

$$f(a;m,\sigma^2),\ \ f(b;m,\sigma^2) \in \mathbb{R}^+, \tag{6}$$

Proof. We have

$$\begin{aligned} I_{X_{a\leftrightarrow b}}(\sigma^2) &= I_{X_{a\leftrightarrow b}}(\theta) = \\ &= \int_a^b \left(\frac{\partial \ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial \sigma^2}\right)^2 f_{a\leftrightarrow b}(x;m,\sigma^2)dx = \\ &= \int_a^b \left(\frac{\partial \ln f_{a\leftrightarrow b}(x;m,\theta)}{\partial \theta}\right)^2 f_{a\leftrightarrow b}(x;m,\theta)dx. \end{aligned}$$

Using (2) and (3), we obtain

$$\begin{aligned}
\ln f_{a\leftrightarrow b}(x;m,\sigma^2) &= -\ln\sqrt{2\pi}-\tfrac{1}{2}\ln\sigma^2-\ln A-\tfrac{1}{2}\left(\tfrac{x-m}{\sigma}\right)^2 = \\
&= -\ln\sqrt{2\pi}-\tfrac{1}{2}\ln\theta-\ln\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)-\Phi\left(\frac{a-m}{\sqrt{\theta}}\right)\right]- \\
&- \tfrac{1}{2}\tfrac{(x-m)}{\theta}^2 = \\
&= \ln f_{a\leftrightarrow b}(x;m,\theta)
\end{aligned}$$

and, it follows

$$\frac{\partial\ln f_{a\leftrightarrow b}(x;m,\theta)}{\partial\theta}=-\frac{1}{2\theta}-\frac{\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)\right]-\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)\right]}{A}+\frac{(x-m)}{2\theta^2}^2.$$

Using the relations

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{b-m}{\sqrt{\theta}}\right)\right] &= \frac{\partial}{\partial\theta}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\frac{b-m}{\sqrt{\theta}}}\exp\{-\frac{1}{2}z^2\}dz\right] = \\
&= -\frac{b-m}{2}\frac{1}{\theta^3}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{b-m}{\sqrt{\theta}}\right)^2\right\} = \\
&= -\frac{b-m}{2}\frac{1}{\theta^2}f(b;m,\theta),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial\theta}\left[\Phi\left(\frac{a-m}{\sqrt{\theta}}\right)\right] &= \frac{\partial}{\partial\theta}\left[\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\frac{a-m}{\sqrt{\theta}}}\exp\{-\frac{1}{2}z^2\}dz\right] = \\
&= -\frac{a-m}{2}\frac{1}{\theta^2}f(a;m,\theta),
\end{aligned}$$

it results

$$\begin{aligned}
\frac{\partial\ln f_{a\leftrightarrow b}(x;m,\theta)}{\partial\theta} &= \frac{\partial\ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial\sigma^2}=\frac{1}{2\sigma^2}\left\{\frac{(x-m)^2}{\sigma^2}+\right. \\
&\left.+\frac{[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)]}{A}-1\right\}
\end{aligned}$$

and Fisher's information will be written

$$I_{X_{a\leftrightarrow b}}(\sigma^2) = \int_a^b \left(\frac{\partial \ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial \sigma^2}\right)^2 f_{a\leftrightarrow b}(x;m,\sigma^2)dx = \frac{1}{4\sigma^8 A}\{I_1+$$

$$+\sigma^4\left[\frac{(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)}{A}-1\right]^2 I_2+$$

$$+2\sigma^2\left[\frac{(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)}{A}-1\right] I_3\right\},$$

where

$$I_2 = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}dx,$$

$$I_3 = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b (x-m)^2 \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}dx,$$

$$I_1 = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b (x-m)^4 \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}dx.$$

By making the change of variables

$$z = \frac{x-m}{\sigma},$$

and, if we consider the formula for integration by parts

$$\int_\alpha^\beta u dv = uv\,|_\alpha^\beta - \int_\alpha^\beta v du,$$

it results

$$I_2 = A = \Phi\left(\frac{b-m}{\sigma}\right) - \Phi\left(\frac{a-m}{\sigma}\right),$$

$$I_3 = -\sigma^2\left\{\left[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)\right]-A\right\},$$

$$I_1 = -\sigma^2\left[(b-m)^3 f(b;m,\sigma^2)-(a-m)^3 f(a;m,\sigma^2)\right]-$$

$$-3\sigma^4\left[(b-m)f(b;m,\sigma^2)-(a-m)f(a;m,\sigma^2)\right]+3\sigma^4 A.$$

Using the final values of the integrals I_2, I_3 and I_1, we obtain (8). □

2. **Invariance of the Fisher information**

Corollary 1. *If* $a = m - \sigma$, $b = m + \sigma$, *then*

$$f_{m-\sigma\leftrightarrow m+\sigma}(x; m, \sigma^2) = \begin{cases} \dfrac{C(m-\sigma, m+\sigma)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \quad if\ m-\sigma \leq x \leq m+\sigma, \\ 0\ if\ x < m-\sigma\ or\ x > m+\sigma, \end{cases} \tag{7}$$

where

$$C(m-\sigma, m+\sigma) = \frac{1}{2\Phi(1) - 1} \approx 2,93 \tag{8}$$

and the Fisher's information measure, relative to the unknown parameter σ^2, *has the value*

$$I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2) \approx 0,03 I_X(\sigma^2). \tag{9}$$

Proof. Using (8), we obtain

$$\begin{aligned} I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2) = & \frac{-2}{4\sigma^4\sqrt{2\pi e}[2\Phi(1)-1]} - \\ & - \frac{3}{4\sigma^4}\left\{\frac{2}{\sqrt{2\pi e}[2\Phi(1)-1]} - 1\right\} - \\ & - \frac{1}{4\sigma^4}\left\{\frac{2}{\sqrt{2\pi e}[2\Phi(1)-1]} - 1\right\}^2 = \\ & = -\frac{1}{4\sigma^4}\left\{-3 + \frac{4}{\sqrt{2\pi e}[\Phi(1)-0,5]} + \right. \\ & \left. + \left(\frac{1}{\sqrt{2\pi e}[\Phi(1)-0,5]} - 1\right)^2\right\} = \\ & = -\frac{1}{4\sigma^4}\{-3 + 2,86 + 0,08\} = \frac{0,03}{2\sigma^4}. \end{aligned}$$

□

Corollary 2. *(Invariance of the Fisher information - the first form) If $a = m$, $b = m + \sigma$ or if $a = m - \sigma$, $b = m$, then*

$$f_{m\leftrightarrow m+\sigma}(x; m, \sigma^2) = \begin{cases} \dfrac{C(m, m+\sigma)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \qquad if\ m \le x \le m+\sigma, \\ 0\ if\ x < m\ or\ x > m+\sigma, \end{cases} \tag{10}$$

and

$$f_{m-\sigma\leftrightarrow m}(x; m, \sigma^2) = \begin{cases} \dfrac{C(m-\sigma, m)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \qquad if\ m-\sigma \le x \le m, \\ 0\ if\ x < m-\sigma\ or\ x > m, \end{cases} \tag{11}$$

where

$$C(m, m+\sigma) = \frac{1}{\Phi(1) - \Phi(0)} = C(m-\sigma, m) = \frac{1}{\Phi(0) - \Phi(-1)} \approx 2,93 \tag{12}$$

and the Fisher's information measures relative to the unknown parameter σ^2 has the same value, namely

$$I_{X_{m\leftrightarrow m+\sigma}}(\sigma^2) = I_{X_{m-\sigma\leftrightarrow m}}(\sigma^2) = \frac{0,03}{2\sigma^4} = I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2). \tag{13}$$

Remark 1. *If we consider the normal variable*

$$Y = X - m, \tag{14}$$

then $E(Y) = 0$, $Var(Y) = Var(X) = \sigma^2$ and the probability density function has the form

$$f_Y(y; \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}, y \in \mathbb{R}. \tag{15}$$

In this case, the random variable $Y_{a\leftrightarrow b}$ has a bilateral truncated normal distribution:

$$f_{a\leftrightarrow b}(y; \sigma^2) = \begin{cases} \frac{C_0(a,b)}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} & if\ a \le y \le b \\ 0 & if\ y < a\ or\ y > b, \end{cases} \tag{16}$$

where

$$C_0 = C_0(a, b) = \frac{1}{[\Phi\left(\frac{b}{\sigma}\right) - \Phi\left(\frac{a}{\sigma}\right)]}. \tag{17}$$

Using (8), the Fisher's information measure, $I_{Y_{a\leftrightarrow b}}(\sigma^2)$ *relative to the unknown parameter* σ^2*, can be written like*

$$I_{Y_{a\leftrightarrow b}}(\sigma^2) = \frac{a^3 f_Y(a;\sigma^2) - b^3 f_Y(b;\sigma^2)}{4\sigma^6 C_0} -$$
$$-\frac{3}{4\sigma^4}\left[\frac{b f_Y(b;\sigma^2) - a f_Y(a;\sigma^2}{C_0} - 1\right] -$$
$$-\frac{1}{4\sigma^4}\left[\frac{b f_Y(b;\sigma^2) - a f_Y(a;\sigma^2}{C_0} - 1\right]^2. \tag{18}$$

Corollary 3. *(Invariance of the Fisher information - the second form)*

$$I_{X_{-\sigma\leftrightarrow\sigma}}(\sigma^2) = I_{X_{m-\sigma\leftrightarrow m+\sigma}}(\sigma^2) = I_{X_{m\leftrightarrow m+\sigma}}(\sigma^2) = I_{X_{m-\sigma\leftrightarrow m}}(\sigma^2) \approx$$
$$\approx \frac{0,03}{2\sigma^4} = 0,03 I_X(\sigma^2). \tag{19}$$

Theorem 2. *(Invariance of the Fisher information - the third form) If* $a = m - k\sigma$, $b = m + k\sigma$*, or* $a = -k\sigma$, $b = k\sigma$*, then the probability density function, denoted by* $f_{a\leftrightarrow b}(x; m, \sigma^2)$*, in(2), has the form*

$$f_{m-k\sigma\leftrightarrow m+k\sigma}(x; m, \sigma^2) = \begin{cases} \dfrac{C(m-k\sigma, m+k\sigma)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} \\ \quad if\ m - k\sigma \le x \le m + k\sigma, \\ 0\ if\ x < m - k\sigma\ or\ x > m + k\sigma, \end{cases} \tag{20}$$

or the form

$$f_{-k\sigma\leftrightarrow k\sigma}(x; m, \sigma^2) = \begin{cases} \dfrac{C(-k\sigma, k\sigma)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x}{\sigma}\right)^2\right\}\ if\ -k\sigma \le x \le k\sigma, \\ 0\ if\ x < -k\sigma\ or\ x > k\sigma, \end{cases} \tag{21}$$

where

$$C(m-k\sigma, m+k\sigma) = C(-k\sigma, k\sigma) = \frac{1}{2\Phi(k) - 1}, k \in \mathbb{N}^*. \tag{22}$$

The Fisher's information measures, relative to the unknown parameter σ^2*, have the same values*

$$I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(\sigma^2) = I_{X_{-k\sigma\leftrightarrow k\sigma}}(\sigma^2) = -\frac{1}{4\sigma^4}\left\{-3 + \frac{(k^3+3k)}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k) - 0,5]} + \right.$$
$$\left. + \left(\frac{k}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k) - 0,5]} - 1\right)^2\right\}, k \in \mathbb{N}^*. \tag{23}$$

Proof. Indeed, using the relations (2), (3) as well as the Theorem 1 and the Corollary 3, we obtain just the above value. □

Corollary 4. *(Invariance of the Fisher information - extended form)*

$$I_{X_{m\leftrightarrow m+k\sigma}}(\sigma^2) = I_{X_{m-k\sigma\leftrightarrow m}}(\sigma^2) = I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(\sigma^2) = I_{X_{-k\sigma\leftrightarrow k\sigma}}(\sigma^2) =$$

$$= -\frac{1}{4\sigma^4}\left\{-3 + \frac{(k^3+3k)}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k)-0,5]} + \left(\frac{k}{\sqrt{2\pi}e^{\frac{k^2}{2}}[\Phi(k)-0,5]} - 1\right)^2\right\}, k \in \mathbb{N}^*. \tag{24}$$

Remark 2. *Using Theorem 2 we obtain :*

a) for $k = 1$, from (23) it results (19).

b) for $k = 2$, from (23) it results

$$I_{X_{m-2\sigma\leftrightarrow m+2\sigma}}(\sigma^2) = I_{X_{-2\sigma\leftrightarrow 2\sigma}}\ (\sigma^2) =$$

$$= -\frac{1}{4\sigma^4}\left\{-3 + \frac{14}{\sqrt{2\pi}e^2[\Phi(2)-0,5]} + \right.$$

$$\left. + \left(\frac{3}{\sqrt{2\pi}e^2[\Phi(2)-0,5]} - 1\right)^2\right\} \approx \tag{25}$$

$$\approx -\frac{1}{4\sigma^4}(-3 + 1,6 + 0,60) =$$

$$= \frac{0,40}{2\sigma^4} = 0,40 I_X(\sigma^2). \tag{26}$$

c) for $k = 3$ we obtain

$$I_{X_{m-3\sigma\leftrightarrow m+3\sigma}}(\sigma^2) = I_{X_{-3\sigma\leftrightarrow 3\sigma}}\ (\sigma^2) =$$

$$= -\frac{1}{4\sigma^4}\left\{-3 + \frac{36}{\sqrt{2\pi e}e^4[\Phi(3)-0,5]} + \right.$$

$$\left. + \left(\frac{2}{\sqrt{2\pi e}e^4[\Phi(3)-0,5]} - 1\right)^2\right\} \approx \tag{27}$$

$$\approx -\frac{1}{4\sigma^4}(-3 + 0,33 + 0,95) = \frac{0,86}{2\sigma^4} = 0,86 I_X(\sigma^2). \tag{28}$$

Conclusion 1. *The invariance properties of the Fisher information, relative to the unknown parameter σ^2, take place then when the normal variable X is truncated on intervals of the forms:*

$$[m - k\sigma, m + k\sigma], [m, m + k\sigma], [m - k\sigma, m], [-k\sigma, k\sigma], k \in \mathbb{N}^*. \tag{29}$$

References

[1] Fătu, C. I., *On the invariance property of the Fisher information*, Studia Univ. "Babeş-Bolyai", Mathematica,Vol. XLIX, No. 4, Dec. 2004, 67-77.

[2] Mihoc, I., Fătu, C. I., *Fisher's Information Measures for the Truncated Normal Distribution (I)*, Analysis, Functional Equations, Approximation and Convexity, Proceedings of the Conference Held in Honor Professor Elena Popoviciu on the Occasion of the 75^{th}Birthday, Editura Carpatica 1999, 171-182.

[3] Mihoc, I., Fătu, C. I., *Fisher's Information Measures and Truncated Normal Distributions (II)*, Seminarul de teoria celei mai bune aproximări, convexitate şi optimizare, 1960-2000, ,,Omagiu memoriei academicianului Tiberiu Popoviciu la 25 de ani de la moarte", 26-29 octombrie, 2000, 153-155.

[4] Mihoc, I., Fătu, C. I., *Calculul probabilităţilor şi statistică matematică*, Casa de Editură Transilvania Press, Cluj-Napoca, 2003.

[5] Rao, C. R., *Liniar Statistical Inference and Its Applications*, John Wiley and Sons, Inc., New York, 1965.

[6] Rényi, A., *Probability Theory*, Akadémiai Kiado, Budapest, 1970.

[7] Schervish, M. J., *Theory of Statistics*, Springer-Verlag New York, Inc. 1995.

Faculty of Economical Sciences at Christian University,
"Dimitrie Cantemir", 3400, Cluj-Napoca, Romania
E-mail address: cfatu@cantemir.cluj.astral.ro