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THE P-LAPLACIAN OPERATOR ON THE SOBOLEV SPACE $W^{1,p}(\Omega)$

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Abstract. In this paper p-Laplacian operator is defined on $W^{1,p}(\Omega)$ in connection with the duality mapping of $W^{1,p}(\Omega)$.

1. **Introduction and preliminary results**

Let Ω be an open bounded subset in $\mathbf{R}^N$, $\mathbf{N} \geq 2$, with smooth boundary and $1 < p < \infty$.

We shall use the standard notations:

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), i = \overline{1, N} \right\},$$

equipped with the norm

$$\|u\|_{1,p}^p = \|u\|_{0,p}^p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{0,p}^p,$$

where $\|\cdot\|_{0,p}$ is the usual norm on $L^p(\Omega)$.

It is well known that $\left(W^{1,p}(\Omega), \|\cdot\|_{1,p}\right)$ is separable, reflexive and uniformly convex Banach space (see e.g. [1], theorem 3.5).

If $u \in W^{1,p}(\Omega)$ we can speak about $u|_{\partial\Omega}$ in the sense of the trace: there is a unique linear and continuous operator $\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega)$ such that γ is surjective and for $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ we have $\gamma u = u|_{\partial\Omega}$.

Then the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$ is

$$W_0^{1,p}(\Omega) = \left\{ u \in W^{1,p}(\Omega) : u|_{\partial\Omega} = 0 \right\} = Ker\gamma.$$

The dual space $(W_0^{1,p}(\Omega))^*$ will be denoted by $W^{-1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

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For each $u \in W^{1,p}(\Omega)$ we put

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N} \right) \quad , \quad |\nabla u| = \left(\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}$$

and let us remark that

$$|\nabla u| \in L^p(\Omega) \quad , \quad |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \in L^{p'}(\Omega) \; , \quad i = \overline{1, N}.$$

By the Poincaré inequality

$$\|u\|_{0,p} \leq const\,(\Omega, N)\, ||\nabla u||_{0,p} \text{ , for all } u \in W_0^{1,p}(\Omega) \text{ ,}$$

the functional

$$W_0^{1,p}(\Omega) \ni u \to \|u\|_{1,p} := ||\nabla u||_{0,p}$$

is a norm on $W_0^{1,p}(\Omega)$, equivalent with $\|\cdot\|_{W^{1,p}(\Omega)}$.

The p-Laplacian operator $\Delta_p u = div \left(|\nabla u|^{p-2} \nabla u \right)$ may be action (see [2] or [6]) from $W_0^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ by

$$< -\Delta_p u, v > \; = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \text{ , for } u, v \in W_0^{1,p}(\Omega).$$

Now we define the p-Laplacian operator on the space $W^{1,p}(\Omega)$.

We define a new equivalent norm on the space $W^{1,p}(\Omega)$:

$$|||u|||_{1,p}^p = ||u||_{0,p}^p + \|\nabla u\|_{0,p}^p = \int_\Omega |u|^p + \int_\Omega \left(\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{p}{2}} .$$

The space $\left(W^{1,p}(\Omega), |||\cdot|||_{1,p}\right)$ is separable, reflexive and uniformly convex Banach space (see [5]).

The dual norm on $(W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$ is denoted by $|||\cdot|||_*$.

If $u \in W^{1,p}(\Omega)$ and $div \left(|\nabla u|^{p-2} \nabla u \right) \in L^{p'}(\Omega)$ we can speak about $|\nabla u|^{p-2} \frac{\partial u}{\partial n}\big|_{\partial\Omega}$ and $|\nabla u|^{p-2} \frac{\partial u}{\partial n}\big|_{\partial\Omega} \in W^{-\frac{1}{p'},\, p'}(\partial\Omega)$ is defined (see [5] and [8]) by

$$< |\nabla u|^{p-2} \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}, v|_{\partial\Omega} > \; = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega div \left(|\nabla u|^{p-2} \nabla u \right) v,$$

$$(\forall) \;\; v \in W^{1,p}(\Omega).$$

If $|\nabla u|^{p-2} \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$ it follows that

$$\int_\Omega -div\left(|\nabla u|^{p-2}\,\nabla u\right) v = \int_\Omega |\nabla u|^{p-2}\,\nabla u \nabla v,\ (\forall)\ v \in W^{1,p}(\Omega).$$

Because the integral $\int_\Omega |\nabla u|^{p-2}\,\nabla u\nabla v$ exists for each $u, v \in W^{1,p}(\Omega)$ we define the operator

$$-\Delta_p : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to (W^{1,p}(\Omega), |||\cdot|||_{1,p})^* \text{ by}$$

$$< -\Delta_p u, v >= \int_\Omega |\nabla u|^{p-2}\,\nabla u\nabla v \text{ , for all } u, v \in W^{1,p}(\Omega).$$

Let us remark that if $u \in W^{1,p}(\Omega)$ then $-\Delta_p u \in (W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$.

Indeed, if $u \in W^{1,p}(\Omega)$ the application $W^{1,p}(\Omega) \ni v \to < -\Delta_p u, v >$ is linear and, since for all $v \in W^{1,p}(\Omega)$:

$$\begin{aligned} |< -\Delta_p u, v >| &= \left|\int_\Omega |\nabla u|^{p-2}\,\nabla u\nabla v\right| \le \int_\Omega |\nabla u|^{p-1}\,|\nabla v| \le \\ &\le \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla v|^p\right)^{\frac{1}{p}} \le |||u|||_{1,p}^{p-1}|||v|||_{1,p}, \end{aligned}$$

it follows that $-\Delta_p u \in (W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$.

2. **Basic results concerning the duality mapping**

Let $(X, ||\cdot||)$ be a real Banach space and X^* its dual.

For a multivalued operator $A : X \to \mathcal{P}(X^*)$, the range of A is defined by

$$R(A) = \bigcup_{x\in D(A)} Ax,$$

where $D(A) = \{x \in X : Ax \ne \phi\}$ is the domain of A.

The operator A is said to be monotone if

$$< x_1^* - x_2^*, x_1 - x_2 >\ge 0 \text{ , for all } x_1, x_2 \in D(A) \text{ and}$$

$$x_1^* \in Ax_1,\ x_2^* \in Ax_2.$$

A continuous function $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ is called a normalization function if it is strictly increasing, $\varphi(0) = 0$ and $\varphi(r) \to \infty$ with $r \to \infty$.

By duality mapping corresponding to the normalization function φ, we mean the set valued operator $J_\varphi : X \to \mathcal{P}(X^*)$ defined by

$$J_\varphi x = \{x^* \in X^* :< x^*, x >= \varphi(\|x\|)\|x\| \ , \ \|x^*\| = \varphi(\|x\|)\} \ ,$$

for $x \in X$.

By the Hahn-Banach theorem one has that $D(J_\varphi) = X$.

We need of the following result:

Theorem 2.1. *If φ is a normalization function, then:*

(i) for each $x \in X$, $J_\varphi x$ is a bounded, closed and convex subset of X^*;

(ii) J_φ is monotone:

$$< x_1^* - x_2^*, x_1 - x_2 >\geq (\varphi(\|x_1\|) - \varphi(\|x_2\|))(\|x_1\| - \|x_2\|) \geq 0 \ ,$$

for each $x_1, x_2 \in X$ and $x_1^* \in J_\varphi x_1$, $x_2^* \in J_\varphi x_2$;

(iii) for each $x \in X$, $J_\varphi x = \partial\Phi(x)$, where $\Phi(x) = \int_0^{\|x\|} \varphi(t)\,dt$ and $\partial\Phi : X \to \mathcal{P}(X^*)$ is the subdifferential of Φ in the sense of convex analysis, i.e.

$$\partial\Phi(x) = \{x^* \in X^* : \Phi(y) - \Phi(x) \geq< x^*, y - x > \ , (\forall) \ y \in X\} \, .$$

For proof we refer to Browder [3], Lions [7], Ciorãnescu [4].

Remark 2.1. We recall that a functional $f : X \to \mathbf{R}$ is said to be Gâteaux differentiable at $x \in X$, if there exists $f'(x) \in X^*$ such that

$$\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} =< f'(x), h > \ , \text{for all } h \in X.$$

If the convex function $f : X \to \mathbf{R}$ is Gâteaux differentiable at $x \in X$ then $\partial f(x) = \{f'(x)\}$.

For example, if $X = (W_0^{1,p}(\Omega), ||\cdot||_{1,p})$, $1 < p < \infty$ and $\varphi(t) = t^{p-1}$, then (see e.g. [6] or [7]) the duality mapping J_φ on the space $W_0^{1,p}(\Omega)$ is exactly the p-Laplacian operator $-\Delta_p$,

$$J_\varphi : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega),$$

$$J_\varphi u = -\Delta_p u \ , (\forall) \ u \in W_0^{1,p}(\Omega).$$

The surjectivity of the duality mapping (see [6]) achieves the existence of the $W_0^{1,p}(\Omega)$-solution for the equation $-\Delta_p u = f$, with $f \in W^{-1,p'}(\Omega)$.

3. The main result

In the sequel, $W^{1,p}(\Omega)$ will be endowed with the norm $|||\cdot|||_{1,p}$.

Theorem 3.1. *The duality mapping on the space $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$, corresponding to the normalization function $\varphi(t) = t^{p-1}$, $1 < p < \infty$, is the single-valued map*

$$J_\varphi : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to (W^{1,p}(\Omega), |||\cdot|||_{1,p})^*$$

$$J_\varphi u = -\Delta_p u + |u|^{p-2}\, u \ , \text{ for each } u \in W^{1,p}(\Omega)\ ,$$

where $-\Delta_p$ is the p-Laplacian operator on the space $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$.

Proof. By the theorem 2.1. $J_\varphi u = \partial\Phi\,(u)$, $(\forall)\ u \in W^{1,p}(\Omega)$, where $\Phi : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to \mathbf{R}$, $\Phi\,(u) = \int_0^{|||u|||_{1,p}} \varphi\,(t)\,dt = \frac{1}{p}|||u|||_{1,p}^p = \frac{1}{p}\,\|u\|_{0,p}^p +$ $+\frac{1}{p}\,\|\nabla u\|_{0,p}^p$ and $\partial\Phi : (W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to \mathcal{P}\left((W^{1,p}(\Omega), |||\cdot|||_{1,p})^*\right)$ is the subdifferential in the sense of convex analysis.

We define the functionals

$$\tilde{\Phi}_1 \ : \ L^p(\Omega) \to \mathbf{R},\ \tilde{\Phi}_1(u) = \frac{1}{p}\,\|u\|_{0,p}^p \ = \frac{1}{p}\int_\Omega |u|^p$$

$$\Phi_2 \ : \ W^{1,p}(\Omega) \to \mathbf{R},\ \Phi_2(u) = \frac{1}{p}\,\|\nabla u\|_{0,p}^p = \frac{1}{p}\int_\Omega |\nabla u|^p$$

and $\Phi_1 : W^{1,p}(\Omega) \to \mathbf{R}$, $\Phi_1 = \tilde{\Phi}_1 / W^{1,p}(\Omega)$

The functional $\tilde{\Phi}_1$ is Gâteaux differentiable (see [9]) and

$$< \tilde{\Phi}_1'(u), v >=< |u|^{p-1}\, sgn\, u, v >, \text{ for all } u, v \in L^p\,(\Omega)\,.$$

By the imbedding $(W^{1,p}(\Omega), |||\cdot|||_{1,p}) \to \left(L^p\,(\Omega)\,, \|\cdot\|_{0,p}\right)$ we have that Φ_1 is Gâteaux differentiable on $(W^{1,p}(\Omega), |||\cdot|||_{1,p})$.

Let the operator $P : W^{1,p}(\Omega) \to L^p\,(\Omega)$ be defined by $P(u) = |\nabla u|$.

If $u \in W^{1,p}(\Omega), u = 0$, then $< \Phi_2'(0), v >= 0$, $(\forall)\ v \in W^{1,p}(\Omega)$.

If $u \neq 0$, a simple computation shows that

$$< P'(u), v >= \frac{\nabla u \cdot \nabla v}{|\nabla u|}\ ,\ (\forall)\ \ v \in W^{1,p}(\Omega).$$

Since the functional $W^{1,p}(\Omega) \ni v \to < P'(u), v >$ is linear and

$$|< P'(u), v >| = \left| \int_\Omega \frac{\nabla u \cdot \nabla v}{|\nabla u|} \right| \leq \int_\Omega |\nabla v| \leq (meas\ \Omega)^{\frac{1}{p'}} \left(\int_\Omega |\nabla v|^p \right)^{\frac{1}{p}} \leq$$

$\leq c|||v|||_{1,p}$, where $c = (meas\ \Omega)^{\frac{1}{p'}}$, it follows that the operator P is Gâteaux differentiable at u.

Since $\Phi_2 = \tilde{\Phi}_1 o P$ one has that the functional Φ_2 is Gâteaux differentiable at u and

$$\begin{aligned} < \quad & \Phi_2'(u), v >=< \tilde{\Phi}_1'(Pu), < P'(u), v >>= \\ = \quad & < |\nabla u|^{p-1}, \frac{\nabla u \cdot \nabla v}{|\nabla u|} >= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v =< -\Delta_p u, v >, (\forall)\ v \in W^{1,p}(\Omega). \end{aligned}$$

Consequently, the functional $\Phi = \Phi_1 + \Phi_2$ is Gâteaux differentiable on the space $W^{1,p}(\Omega)$ and

$$< \Phi_1'(u), v >=< -\Delta_p u + |u|^{p-2} u, v >, (\forall)\ u, v \in W^{1,p}(\Omega).$$

Using the convexity of the functional Φ, by remark 2.1 it follows that $J_\varphi u = \Phi'(u) = -\Delta_p u + |u|^{p-2} u$, for all $u \in W^{1,p}(\Omega)$. □

Remark 3.1. By the theorem 2.1 we have

$$|||J_\varphi u|||_* = \varphi\left(|||u|||_{1,p}\right) = |||u|||_{1,p}^{p-1},$$

where $||| \cdot |||_*$ is dual norm of $||| \cdot |||_{1,p}$.

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