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# PROJECTORS AND HALL $\pi$-SUBGROUPS IN FINITE $\pi$-SOLVABLE GROUPS

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**Abstract**. Let $\pi$ be a set of primes and $\underline{X}$ be a $\pi$-closed Schunck class with the $P$ property. The paper gives conditions with respect to which an $\underline{X}$-projector $H$ of a finite $\pi$-solvable group $G$ is an Hall $\pi$-subgroup of $G$, and consequently we have that $N_G(N_G(H)) = N_G(H)$.

## 1. **Preliminaries**

All groups considered in the paper are finite. Let $\pi$ be a set of primes, $\pi'$ the complement to $\pi$ in the set of all primes and $O_{\pi'}(G)$ the largest normal $\pi'$-subgroup of a group $G$.

We first give some useful definitions.

**Definition 1.1.** ([8], [11]) a) A class $\underline{X}$ of groups is a *homomorph* if $\underline{X}$ is epimorphically closed, i.e. if $G \in \underline{X}$ and $N$ is a normal subgroup of $G$, then $G/N \in \underline{X}$.

b) A group $G$ is *primitive* if $G$ has a *stabilizer*, i.e. a maximal subgroup $H$ with $core_G H = \{1\}$, where $core_G H = \cap\{H^g/g \in G\}$.

c) A homomorph $\underline{X}$ is a *Schunck class* if $\underline{X}$ is *primitively closed*, i.e. if any group $G$, all of whose primitive factor groups are in $\underline{X}$, is itself in $\underline{X}$.

**Definition 1.2.** a) A positive integer $n$ is said to be a $\pi$*-number* if for any prime divisor $p$ of $n$ we have $p \in \pi$.

b) A finite group $G$ is a $\pi$*-group* if $|G|$ is a $\pi$-number.

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**Definition 1.3.** ([6]) A group $G$ is *$\pi$-solvable* if every chief factor of $G$ is either a solvable $\pi$-group or a $\pi'$-group. For $\pi$ the set of all primes, we obtain the notion of *solvable group*.

**Definition 1.4.** A class $\underline{X}$ of groups is said to be *$\pi$-closed* if

$$G/O_{\pi'}(G) \in \underline{X} \;\Rightarrow\; G \in \underline{X}.$$

A $\pi$-closed homomorph, respectively a $\pi$-closed Schunck class is called *$\pi$-homomorph*, respectively *$\pi$-Schunck class*.

**Definition 1.5.** ([7], [8]) Let $\underline{X}$ be a class of groups, $G$ a group and $H$ a subgroup of $G$.

a) $H$ is an *$\underline{X}$-maximal subgroup* of $G$ if: (i) $H \in \underline{X}$; (ii) $H \leq H^* \leq G$, $H^* \in \underline{X}$ imply $H = H^*$.

b) $H$ is an *$\underline{X}$-projector* of $G$ if, for any normal subgroup $N$ of $G$, $HN/N$ is $\underline{X}$-maximal in $G/N$.

c) $H$ is an *$\underline{X}$-covering subgroup* of $G$ if: (i) $H \in \underline{X}$; (ii) $H \leq K \leq G$, $K_0 \trianglelefteq K$, $K/K_0 \in \underline{X}$ imply $K = HK_0$.

**Definition 1.6.** ([3], [4]) Let $\underline{X}$ be a class of groups. We say that $\underline{X}$ has the *$P$ property* if, for any $\pi$-solvable group $G$ and for any minimal normal subgroup $M$ of $G$ such that $M$ is a $\pi'$-group, we have $G/M \in \underline{X}$.

The following results are used in this paper.

**Theorem 1.7.** ([1]) *A solvable minimal normal subgroup of a group is abelian.*

**Theorem 1.8.** ([1]) *Suppose that $G$ has a $\neq \{1\}$ normal solvable subgroup and let $S$ be a maximal subgroup of $G$ with $core_G S = \{1\}$. Then, the existence of a $\neq \{1\}$ normal solvable subgroup of $S$ implies the existence of a normal subgroup $N \neq \{1\}$ of $S$ with $(|N|, |G : S|) = 1$.*

**Theorem 1.9.** ([2]) *a) Let $\underline{X}$ be a class of groups, $G$ a group and $H$ a subgroup of $G$. If $H$ is an $\underline{X}$-covering subgroup of $G$ or $H$ is an $\underline{X}$-projector of $G$, then $H$ is $\underline{X}$-maximal in $G$.*

*b) If $\underline{X}$ is a homomorph and $G$ is a group, then a subgroup $H$ of $G$ is an $\underline{X}$-covering subgroup of $G$ if and only if $H$ is an $\underline{X}$-projector in any subgroup $K$ of $G$ with $H \subseteq K$.*

**Theorem 1.10.** *Let $\underline{X}$ be a homomorph.*

*a) ([7]) If $H$ is an $\underline{X}$-covering subgroup of a group $G$ and $N$ is a normal subgroup of $G$, then $HN/N$ is an $\underline{X}$-covering subgroup of $G/N$.*

*b) ([8]) If $H$ is an $\underline{X}$-projector of a group $G$ and $N$ is a normal subgroup of $G$, then $HN/N$ is an $\underline{X}$-projector of $G/N$.*

*c) ([7]) If $H$ is an $\underline{X}$-covering subgroup of $G$ and $H \leq K \leq G$, then $H$ is an $\underline{X}$-covering subgroup of $K$.*

**Theorem 1.11.** ([5]) *Let $\underline{X}$ be a $\pi$-homomorph. The following conditions are equivalent:*

*(1) $\underline{X}$ is a Schunck class;*

*(2) any $\pi$-solvable group has $\underline{X}$-covering subgroups;*

*(3) any $\pi$-solvable group has $\underline{X}$-projectors.*

## 2. **Hall $\pi$-subgroups in finite $\pi$-solvable groups**

Of special interest in this paper will be the Hall $\pi$-subgroups and some of their properties. The Hall subgroups were given in [9]. Ph. Hall studied them in finite solvable groups. In [6], S. A. Čunihin extended this study to finite $\pi$-solvable groups.

**Definition 2.1.** Let $G$ be a group and $H$ a subgroup of $G$.

a) $H$ is a *$\pi$-subgroup* of $G$ if $H$ is a $\pi$-group.

b) $H$ is an *Hall $\pi$-subgroup* of $G$ if: (i) $H$ is a $\pi$-subgroup of $G$; (ii) $(|H|, |G : H|) = 1$, i.e. $|G : H|$ is a $\pi'$-number.

We shall use some properties of the Hall $\pi$-subgroups given in [10]:

**Theorem 2.2.** ([10]) (Ph. Hall, S. A. Čunihin) *If $G$ is a $\pi$-solvable group, then:*

*a) $G$ has Hall $\pi$-subgroups and $G$ has Hall $\pi'$-subgroups;*

*b) any two Hall $\pi$-subgroups of $G$ are conjugate in $G$; any two Hall $\pi'$-subgroups of $G$ are conjugate in $G$ too.*

**Theorem 2.3.** ([10]) *Let $G$ be a group and $H$ an Hall $\pi$-subgroup of $G$.*

*a) If $H \leq K \leq G$, then $H$ is an Hall $\pi$-subgroup of $K$.*

*b) If $N$ is a normal subgroup of $G$, then $HN/N$ is an Hall $\pi$-subgroup of $G/N$.*

We complete these properties with two new ones, which will be used in the formation theory considerations in the main section of this paper.

**Theorem 2.4.** *Let $G$ be a $\pi$-solvable group, $H$ a subgroup of $G$ and $N$ a normal subgroup of $G$. If $HN/N$ is an Hall $\pi$-subgroup of $G/N$ and $H$ is an Hall $\pi$-subgroup of $HN$, then $H$ is an Hall $\pi$-subgroup of $G$.*

**Proof.** (i) $H$ is a $\pi$-subgroup of $G$, since $H$ is a $\pi$-subgroup of $HN$.

(ii) We shall prove that $|G : H|$ is a $\pi'$-number. Indeed, we know that $|G : HN| = |G/N : HN/N|$ is a $\pi'$-number. Further, $|HN : H|$ is a $\pi'$-number too. Then $|G : H| = |G : HN||HN : H|$ is a $\pi'$-number. □

**Theorem 2.5.** *If $G$ is a $\pi$-solvable group and $H$ is a Hall $\pi$-subgroup of $G$, then $N_G(N_G(H)) = N_G(H)$.*

**Proof.** We know that

$$N_G(H) = \{g \in G / H^g = H\} \supseteq H$$

and so we have $N_G(H) \subseteq N_G(N_G(H))$. We now prove that $N_G(N_G(H)) \subseteq N_G(H)$. Let $x \in N_G(N_G(H))$. It is known that $N_G(H) \trianglelefteq N_G(N_G(H))$. It follows that $N_G(H)^x = N_G(H)$, hence $H^x \subseteq N_G(H)^x = N_G(H)$, which implies by 2.3.a) that $H$ and $H^x$ are Hall $\pi$-subgroups of $N_G(H)$. Applying Hall-Čunihin Theorem 2.2.b), we obtain that $H$ and $H^x$ are conjugate in $N_G(H)$. So there is an element $y \in N_G(H)$ such that $(H^x)^y = H$. It follows that $H^{xy} = H$, hence $xy \in N_G(H)$. But $y \in N_G(H)$ implies $y^{-1} \in N_G(H)$ and so $x = (xy)y^{-1} \in N_G(H)$. □

## 3. Projectors which are Hall $\pi$-subgroups in finite $\pi$-solvable groups

In [8], W. Gaschütz gives for finite solvable groups the following result: If $\underline{X}$ is a Schunck class, $G$ a solvable group and $S$ an $\underline{X}$-projector of $G$ such that $S$ is a $p$-group, then $S$ is a Sylow $p$-subgroup of $G$.

It is the aim of this paper to study similar properties in the more general case of finite $\pi$-solvable groups.

All groups considered in this section are finite $\pi$-solvable.

**Theorem 3.1.** *Let $\underline{X}$ be a $\pi$-Schunck class with the $P$ property. If $G$ is a $\pi$-solvable group, such that there is a minimal normal subgroup $M$ of $G$ which is a $\pi'$-group, and if $H$ is an $\underline{X}$-projector of $G$ which is a $\pi$-group, then $H$ is an Hall $\pi$-subgroup of $G$.*

**Proof.** We will show that $|G : H|$ is a $\pi'$-number. Let $M$ be a minimal normal subgroup of $G$, such that $M$ is a $\pi'$-group. We know that $\underline{X}$ has the $P$ property, and so, by 1.6., we have $G/M \in \underline{X}$.

On the other side, $H$ being an $\underline{X}$-projector of $G$, we have, by 1.10., that $HM/M$ is an $\underline{X}$-projector of $G/M$. Now 1.9.a) implies that $HM/M$ is $\underline{X}$-maximal in $G/M$. But $G/M \in \underline{X}$. It follows that $HM/M = G/M$, hence $HM = G$. From this and from $HM/M \cong H/H \cap M$, we obtain that

$$|G : H| = |HM : H| = |M : H \cap M|.$$

Since $|M : H \cap M|$ divides $|M|$ which is a $\pi'$-number, we obtain that $|M : H \cap M|$ is also a $\pi'$-number. Hence $|G : H|$ is a $\pi'$-number. $\square$

In order to renounce to the condition on the group $G$ of having a minimal normal subgroup $M$ which is a $\pi'$-group, the next theorem contains the assumption that $H$ is an $\underline{X}$-covering subgroup of $G$. This means, by 1.9.b), that $H$ is a particular $\underline{X}$-projector.

**Theorem 3.2.** *Let $\underline{X}$ be a $\pi$-Schunck class with the $P$ property. If $G$ is a $\pi$-solvable group and $H$ is an $\underline{X}$-covering subgroup of $G$ which is a $\pi$-group, then $H$ is an Hall $\pi$-subgroup of $G$.*

**Proof.** By induction on $|G|$. We consider two cases:

1) There is a minimal normal subgroup $M$ of $G$, such that $M$ is a $\pi'$-group. By 1.9.b), $H$ is an $\underline{X}$-projector of $G$. Applying theorem 3.1., it follows that $H$ is an Hall $\pi$-subgroup of $G$.

2) Any minimal normal subgroup $M$ of $G$ is a solvable $\pi$-group. Hence, by 1.7., $M$ is abelian. If $H = G$, it follows from $H$ $\pi$-group that $H$ is an Hall $\pi$-subgroup of $G = H$. Let now $H \neq G$. We distinguish two possibilities:

a) For any minimal normal subgroup $M$ of $G$ we have $HM = G$.

Let us first prove that $H$ is a maximal subgroup of $G$. Indeed, we have $H < G$. Further, if $H \leq H^* < G$, we prove that $H = H^*$. Suppose that $H < H^*$, and let $h^* \in H^* \setminus H$. Let $M$ be a minimal normal subgroup of $G$. By the above, we have that $M$ is abelian and $G = HM$. So $h^* = hm$, where $h \in H$, $m \in M$. It follows that $m = h^{-1}h^* \in M \cap H^*$. Let us prove that $M \cap H^* = \{1\}$. Suppose that $M \cap H^* \neq \{1\}$. We have $M \cap H^* \trianglelefteq H^*$. Further, $M \cap H^* \trianglelefteq G$, since if $x \in G = HM = H^*M = MH^*$ and $m \in M \cap H^*$, then $x = m_1h^*$, where $m_1 \in M$, $h^* \in H^*$, and $M$ being abelian, we have:

$$x^{-1}mx = (m_1h^*)^{-1}m(m_1h^*) = (h^*)^{-1}m_1^{-1}mm_1h^* = (h^*)^{-1}mm_1^{-1}m_1h^* =$$

$$= (h^*)^{-1}mh^* \in M \cap H^*.$$

So $M \cap H^* \trianglelefteq G$, $M \cap H^* \subseteq M$, $M \cap H^* \neq \{1\}$. But $M$ is a minimal normal subgroup. Hence $M \cap H^* = M$, which implies that $M \subseteq H^*$ and so $G = H^*M = H^*$, a contradiction with $H^* < G$. It follows that $M \cap H^* = \{1\}$. Hence $m = 1$ and so $h^* = h \in H$, in contradiction with the choice of $h^*$. We proved that $H = H^*$. So $H$ is a maximal subgroup of $G$.

Let us notice that $core_G H = \{1\}$. Indeed, if we suppose that $core_G H \neq \{1\}$, it follows since $core_G H \trianglelefteq G$ that there exists a minimal normal subgroup $M$ of $G$ such that $M \subseteq core_G H$. We obtain $G = HM \subseteq H core_G H = H$, in contradiction with $H \neq G$. So $core_G H = \{1\}$.

We are now in the hypotheses of theorem 1.8.. By 1.8., it follows the existence of a normal subgroup $N \neq \{1\}$ of $H$, such that $(|N|, |G : H|) = 1$. But $H$ being a

$\pi$-group, $N$ is also a $\pi$-group. Then $|G : H|$ is a $\pi'$-number. It follows that $H$ is an Hall $\pi$-subgroup of $G$.

b) There is a minimal normal subgroup $M$ of $G$ such that $HM \neq G$.

We apply the induction to the $\pi$-solvable group $HM$, with $|HM| < |G|$. By 1.10.c), $H$ is an $\underline{X}$-covering subgroup of $HM$. Further, $H$ is a $\pi$-group. By the induction, $H$ is an Hall $\pi$-subgroup of $HM$.

We now apply the induction to the $\pi$-solvable group $G/M$, with $|G/M| < |G|$. By 1.10.a), $HM/M$ is an $\underline{X}$-covering subgroup of $G/M$. Further, we have that $|HM/M| = |H/H \cap M|$ divides $|H|$, and so $HM/M$ is a $\pi$-group. By the induction, $HM/M$ is an Hall $\pi$-subgroup of $G/M$.

Finally, theorem 2.4. leads us to the conclusion that $H$ is an Hall $\pi$-subgroup of $G$. □

**Corollary 3.3.** *Let $\underline{X}$ be a $\pi$-Schunck class with the $P$ property. If $G$ is a $\pi$-solvable group and $H$ is an $X$-covering subgroup of $G$ which is a $\pi$-group, then $N_G(N_G(H)) = N_G(H)$.*

**Proof.** Follows from 3.2. and 2.5.. □

## References

[1] Baer, R., *Classes of finite groups and their properties*, Illinois J. Math., **1**, No.2, 1957, 115-187.

[2] Covaci, R., *Projectors and covering subgroups*, Studia Univ. Babeş-Bolyai, Math., **XXVII**, 1982, 33-36.

[3] Covaci, R., *On $\pi$-Schunck classes with the $P$ property*, Seminar on Algebra, Babeş-Bolyai Univ. Cluj-Napoca, Faculty of Mathematics, Research Seminars, Preprint No.5, 1988, 22-34.

[4] Covaci, R., *$\pi$-Closure and the $P$ property*, Studia Univ. Babeş-Bolyai, Math., **XXXVIII**, No.1, 1993, 15-19.

[5] Covaci, R., *A characterization of $\pi$-closed Schunck classes*, Studia Univ. Babeş-Bolyai, Math., **XLVIII**, No.3, 2003, 63-70.

[6] Čunihin, S. A., *O teoremah tipa Sylowa*, Dokl. Akad. Nauk SSSR, **66**, No.2, 1949, 165-168.

[7] Gascütz, W., *Zur Theorie der endlichen auflösbaren Gruppen*, Math. Z., **80**, No.4, 1963, 300-305.

[8] Gascütz, W., *Selected topics in the theory of soluble groups*, Australian National University, Canberra, 1969.

[9] Hall, Ph., *A note on soluble groups*, J. London Math. Soc., **3**, 1928, 98-105.

[10] Huppert, B., *Endliche Gruppen I*, Berlin - New York, Springer Verlag, 1967.

[11] Schunck, H., $\underline{H}$*-Untergruppen in endlichen auflösbaren Gruppen*, Math. Z., **97**, No.4, 1967, 326-330.

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