STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **L**, Number 1, March 2005

ON SOME APPLICATIONS OF INTERPOLATION OPERATORS

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Abstract. The goal of this paper is to give applications of interpolation operators, with a special emphasis on optimal approximation of some linear functionals and the construction of methods for the solution of equations on $\mathbb{R}$.

Let $\mathcal{B}$ be a linear space of real-valued functions defined on a domain $\Omega \subset \mathbb{R}^n$, $\mathcal{A} \subset \mathcal{B}$ and $\Lambda = \{\lambda_i |\ \lambda_i : \mathcal{B} \to \mathbb{R},\ i = 1, \ldots, N\}$, a set of linear functionals. For $f \in \mathcal{B}$, is denoted by $\Lambda(f) = \{\lambda_i(f) |\ \lambda_i \in \Lambda,\ i = 1, \ldots, N\}$, the informations on f suitable to Λ.

An operator $P : \mathcal{B} \to \mathcal{A}$, for which

$$\lambda_i(Pf) = \lambda_i(f), \quad i = 1, \ldots, N,$$

$f \in \mathcal{B}$, is an interpolation operator that interpolates the set Λ, while

$$f = Pf + Rf$$

is the interpolation formula generated by P, with R the remainder operator.

The number $r \in \mathbb{N}$ for which $Pf = f$, for all $f \in \mathbf{P}_r^n$ and there exists $g \in \mathbf{P}_{r+1}^n$, such that $Pg \neq g$, where $\mathbf{P}_r^n$ is the set of all polynomial functions of the total degree at most r, is the degree of exactness of the operator P.

Received by the editors: 10.01.2005.

2000 *Mathematics Subject Classification.* 41A05, 65D05, 65D30.

Key words and phrases. operator, interpolation, linear functional, equation, optimal.

The purpose of this paper is to discuss applications of interpolation operators to the approximation of some linear functionals, the construction of some homogeneous cubature formulas, the construction of numerical methods for solving of some operatorial equations.

1. **Optimal approximation in sense of Sard**

One suppose here that $\Lambda := \Lambda_B = \{\lambda_{kj} : \ H^{m,2}[a,b] \to \mathbb{R}, \ k = 1, \dots, n, \ j \in I_k\}$, with $I_k \subset \{0, 1, \dots, r_k\}$, $r_k \in \mathbb{N}$, $r_k < m$, is a set of Birkhoff-type functionals, i.e. $\lambda_{kj}(f) = f^{(j)}(x_k)$, $x_k \in [a,b]$, $x_k \neq x_j$ for $k \neq j$. Let $\lambda : H^{m,2}[a,b] \to \mathbb{R}$, be a given linear functional such that the elements of the set $\Lambda_B \cup \{\lambda\}$ to be linear independent. One considers the approximation formula

$$\lambda(f) = \sum_{k=1}^{n} \sum_{j \in I_k} A_{kj} f^{(j)}(x_k) + R_N(f), \tag{1}$$

where $N = |I_1| + \dots |I_n| - 1$.

Definition 1. Formula (1), with prescribed points $x_k \in [a,b]$, $k = 1, \dots, n$, for which:

i) $R_N(e_\nu) = 0, \ \nu = 0, 1, \dots, m-1$

ii) $\displaystyle\int_a^b K_m^2(t)dt \to minim,$

where K is the corresponding Peano kernel:

$$\begin{aligned} K(t) &:= R_N\left(\frac{(\cdot - t)_+^{m-1}}{(m-1)!}\right) \\ &= \lambda\left[\frac{(\cdot - t)_+^{m-1}}{(m-1)!}\right] - \sum_{k=1}^{n}\sum_{j \in I_k} A_{kj} \frac{(x_k - t)_+^{m-j-1}}{(m-j-1)!}, \end{aligned}$$

is called optimal in sense of Sard.

In 1964, I. J. Schoenberg [10] has established a relationship between the optimal approximation of linear operators, in particular, optimality in sense of Sard, and spline interpolation problems.

So, let $S : H^{m,2}[a,b] \to \mathcal{S}_{2m-1}(\Lambda_B)$ be a natural spline interpolation operator of the order $2m-1$, suitable to Λ_B.

Remark 2. [3] If Λ_B contains at least m functionals of Hermite-type then S exists and is unique.

For $f \in H^{m,2}[a,b]$, let

$$f = Sf + Rf$$

be the natural spline interpolation formula generated by S.

It follows [10], that

$$\lambda(f) = \lambda(Sf) + \lambda(Rf) \tag{2}$$

is the formula of the form (1) that is optimal in sense of Sard.

For example, if

$$\lambda(f) = \int_a^b f(x)dx$$

then (2) becomes an optimal, in sense of Sard, quadrature formula.

As an application, let us find the quadrature formula of the form

$$\int_0^1 f(x)dx = A_{00}f(0) + A_{10}F\left(\frac{1}{2}\right) + A_{21}f'\left(\frac{1}{2}\right) + A_{30}d(1) + R(f)$$

that is optimal in sense of Sard. Using, for example, the cubic spline interpolation formula

$$f(x) = s_{00}(x)f(0) + s_{10}(x)f\left(\frac{1}{2}\right) + s_{11}(x)f'\left(\frac{1}{2}\right) + s_{20}(x)f(1) + (Rf)(x)$$

where

$$s_{00}(x) = 1 - 3x + 4x^3 - 4\left(x - \frac{1}{2}\right)_+^3 - 6\left(x - \frac{1}{2}\right)_+^2$$

$$s_{10}(x) = 3x - 4x^3 + 8\left(x - \frac{1}{2}\right)_+^3 - 4(x-1)_+^3$$

$$s_{11}(x) = -\frac{1}{2}x + 2x^3 - 6\left(x - \frac{1}{2}\right)_+^2 - 2(x-1)_+^3$$

$$s_{20}(x) = -4\left(x - \frac{1}{2}\right)_+^3 + 6\left(x - \frac{1}{2}\right)_+^2 + 4(x-1)_+^3$$

and

$$(Rf)(x) = \int_0^1 \varphi_1(x,t)f''(t)dt$$

with

$$\varphi_1(x,t) = (x-t)_+ - s_{10}(x)\left(\frac{1}{2}-t\right)_+ - s_{11}(x)\left(\frac{1}{2}-t\right)_+^0 - s_{20}(x)(1-t)_+$$

it follows that the optimal coefficients A_{ij}^* respectively the optimal kernel K_1^* are

$$A_{ij}^* = \int_0^1 s_{ij}(x)dx$$

and

$$K_1^*(t) = \int_0^1 \varphi_1(x,t)dx.$$

One obtains:

$$A_{00}^* = \frac{3}{16}, \quad A_{10}^* = \frac{5}{8}, \quad A_{11}^* = 0, \quad A_{20}^* = \frac{3}{16}$$

and

$$K_1^*(t) = \frac{(1-t)^2}{2} - \frac{5}{8}\left(\frac{1}{2}-t\right)_+ - \frac{3}{16}(1-t).$$

We also have

$$\int_0^1 (K_1^*(t))^2 dt = \frac{1}{2^{10}\cdot 5}$$

Hence

$$|R(f)| \leq \frac{1}{32\sqrt{5}}\|f''\|_2.$$

Remark 2. The gaussian quadratures are optimal in sense of Sard - all the coefficients and nodes are determined from the condition i). But, the nodes in the gaussian quadrature formula of the form

$$\int_{-1}^1 f(x)dx = \sum_{i=1}^n A_i f(x_i) + R_n(f) \tag{3}$$

are the zeros of the Cebyshev polynomial T_n, i.e.

$$x_k^* = \cos\frac{2k-1}{2n}\pi, \quad k = 1,\ldots,n.$$

It means that the optimal coefficients of the quadrature form (3) are:

$$A_i^* = \int_{-1}^1 l_i^*(x)dx, \quad i = 1,\ldots,n$$

where l_i^* are the fundamental Lagrange interpolation polynomials corresponding to the interpolation nodes x_i^*, $i = 1,\ldots,n$.

2. **Homogeneous cubature formulas**

Let D be a domain in $\mathbb{R}^2$, $f : D \to \mathbb{R}$ an integrable function on D and $\Lambda(f) = \{\lambda_k(f) |\ k = 1, \ldots, N\}$ some informations on f. Next, one suppose that $\lambda_k(f)$, $k = 1, \ldots, N$, are punctual values of f or of certain of its derivatives, i.e. the values of them at the cubature nodes.

One considers the cubature formula

$$I^{xy} := \int\int_D f(x,y)dxdy = \sum_{i=1}^{N} A_i \lambda_i(f) + R_N(f) \tag{4}$$

where A_i, $i = 1, \ldots, N$ are of the cubature coefficients and $R_N(f)$ is the remainder term.

The problem is to find the parameters of such a cubature formula (coefficients and nodes) and to study the corresponding remainder term. Most solutions for this problem has been obtained when D is a regular domain in $\mathbb{R}^2$ (rectangle, triangle, etc.) and the informations are regularly spaced. For example, the product and the boolean sum rules belong to this class of cubature procedures.

So, let $D \subset \mathbb{R}^2$ be a rectangle, $D = [a, b] \times [c, d]$ and $\lambda^x(f) = \{\lambda_i^x(f) |\ i = 0, 1, \ldots, m\}$, $\Lambda^y(f) = \{\lambda_j^y(f) |\ j = 0, 1, \ldots, n\}$, $m, n \in \mathbb{N}$ some given partial information on f, with regard to x respectively y.

One considers the quadrature formulas:

$$I^x f := \int_a^b f(x,y)dx = (Q_1^x f)(\cdot, y) + (R_1^x f)(\cdot, y) \tag{5}$$

and

$$I^y f := \int_c^d f(x,y)dy = (Q_1^y f)(x, \cdot) + (R_1^y f)(x, \cdot) \tag{6}$$

where

$$(Q_1^x f)(\cdot, y) = \sum_{i=0}^{m} A_i \lambda_i^x(f)$$

$$(Q_1^y f)(x, \cdot) = \sum_{j=0}^{n} B_j \lambda_j^y(f)$$

and R_1^x, R_1^y the corresponding remainder operators:

$$R_1^x = I^x - Q_1^x$$

$$R_1^y = I^y - Q_1^y.$$

We have the following decompositions of the double integral operator I^{xy}:

$$I^{xy} = Q_1^x Q_1^y + (I^y R_1^x + I^x R_1^y - R_1^x R_1^y) \tag{7}$$

and

$$I^{xy} = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) + R_1^x R_1^y \tag{8}$$

The identities (7) and (8) generate so called product cubature formula

$$I^{xy} f = (Q_1^x Q_1^y) f + (R_1^x I^y + I^x R_1^y - R_1^x R_1^y) f$$

respectively, the boolean-sum cubature formula

$$I^{xy} f = (Q_1^x I^y + I^x Q_1^y - Q_1^x Q_1^y) f + R_1^x R_1^y f.$$

Now, if p_1 and q_1 are the approximation orders of Q_1^x respectively Q_1^y ($ord(Q_1^x) = p_1$, $ord(Q_1^y) = q_1$), it follows that the order of the product cubature formula is $\min\{p_1, q_1\}$, while the order of boolean-sum cubature formula is $p_1 + q_1$ [5].

Hence, the boolean-sum cubature rule has the remarkable property that it has a high approximation order. Otherwise, the boolean-sum cubature formula contains the simple integrals $I^x f$ and $I^y f$. But, this simple integrals can be approximated, in a second level of approximation, using new quadrature procedures, i.e.

$$I^x f = Q_2^x f + R_2^x f$$

respectively

$$I^y f = Q_2^y f + R_2^y f.$$

This way, from (8), is obtained

$$I^{xy} = Q^{xy} + R^{xy} \tag{9}$$

where

$$Q^{xy} = Q_1^x Q_2^y + Q_2^x Q_1^y - Q_1^x Q_1^y \tag{10}$$

and

$$R^{xy} = Q_1^x R_2^y + Q_1^y R_2^x + R_1^x R_1^y \tag{11}$$

As can be seen, from (11) follows

$$ord(Q^{xy}) = \min\{ord(Q_1^x) + ord(Q_1^y), ord(Q_2^x) + 1, ord(Q_2^y) + 1\}$$

If

$$ord(Q_2^x) = ord(Q_1^x) + ord(Q_1^x) - 1$$
$$ord(Q_2^y) = ord(Q_1^y) + ord(Q_1^y) - 1$$

then the cubature formula given by (9) is called a homogeneous cubature formula [5].

One of the main procedure to construct homogeneous cubature formulas is based on the interpolation formulas. It is well known that each interpolation formula give rise to a quadrature or cubature formula.

Remark 3. [6] If the multivariate interpolation formula is a homogeneous one, then the suitable cubature formula is also an homogeneous cubature formula.

To illustrate it, we give some simple examples:

Example 2. Let $f : D_h \to \mathbb{R}$, with $D_h = [0, b] \times [0, b]$ be given and $\Lambda(f) = \{f(0,0), f(h,0), f(0,h), f(h,h)\}$. For the partial informations on f: $\Lambda_1^x(f) = \{f(0,y), f(h,y)\}$ respectively $\Lambda_1^y(f) = \{f(x,0), f(x,h)\}$, one considers the Lagrange's operators L_1^x and L_1^y:

$$(L_1^x f)(x,y) = \frac{h-x}{h} f(0,y) + \frac{x}{h} f(h,y),$$
$$(L_1^y f)(x,y) = \frac{h-y}{h} f(x,0) + \frac{y}{h} f(x,h).$$

If $R_1^x = I - L_1^x$ and $R_1^y = I - L_1^y$, with I the identical operator, then we have

$$I = L_1^x \oplus L_1^y + R_1^x R_1^y, \tag{12}$$

the boolean sum decomposition of the identity operator. Also, we have

$$f = L_1^x \oplus L_1^y f + R_1^x R_1^y f$$

or

$$f = (L_1^x + L_1^y - L_1^x L_1^y) f + R_1^x R_1^y f$$

Now, if $L_1^x f$ and $L_1^y f$ are interpolated, in a second level, by the Hermite's operators H_3^y respectively H_3^x suitable to the information sets

$$\Lambda_2^y(f) = \{f(x,0), f^{(0,1)}(x,0), f(x,h), f^{(0,1)}(x,h)\}$$

and

$$\Lambda_2^x(f) = \{f(0,y), f^{(1,0)}(0,y), f(h,y), f^{(1,0)}(h,y)\},$$

one obtains

$$f = (L_1^x H_3^y + H_3^x L_1^y - L_1^x L_1^y)f + (L_1^x R_3^y + L_1^y R_3^x + R_1^x R_1^y)f, \tag{13}$$

where $R_3^x = I - H_3^x$ and $R_3^y = I - H_3^y$. Taking into account that

$$ord(L_1^x L_1^y) = ord(H_3^x) = ord(H_3^y) = 4$$

(13) is a homogeneous interpolation formula of order 4 [4].

Theorem 4. *The cubature formula*

$$\int\int_{D_h} f(x,y)dxdy = Q(f) + R(f)$$

where

$$Q(f) = \int\int_{D_h} ((L_1^x H_3^y + H_3^x L_1^y - L_1^x L_1^y)f)(x,y)dxdy$$

or

$$Q = Q_1^x Q_3^y + Q_3^x Q_1^y + Q_1^x Q_1^y$$

and

$$R(f) = \int\int_{D_h} ((L_1^x R_3^y + L_1^y R_3^x + R_1^x R_1^y)f)(x,y)dxdy$$

is a homogeneous cubature formula of order 6.

Proof. Suppose that $f \in C^{4,4}(D_h)$. Then

$$(R_1^x f)(x,y) = \frac{x(x-h)}{2} f^{(2,0)}(\xi, y)$$

$$(R_1^y f)(x,y) = \frac{y(y-h)}{2} f^{(0,2)}(x, \eta_1)$$

$$(R_3^x f)(x,y) = \frac{x^2(x-h)^2}{24} f^{(4,0)}(\xi_2, y)$$

$$(R_3^y f)(x,y) = \frac{y^2(y-h)^2}{24} f^{(0,4)}(x,\eta_2)$$

with $\xi_1, \xi_2, \eta_1, \eta_2 \in [0,h]$.

As

$$\int_0^h (R_1^x f)(x,y)dx = \frac{h^3}{12} f^{(2,0)}(\mu_1,y)$$

$$\int_0^h (R_1^y f)(x,y)dy = \frac{h^3}{12} f^{(0,2)}(x,\nu_1)$$

$$\int_0^h (R_3^x f)(x,y)dx = \frac{h^5}{720} f^{(4,0)}(\mu_2,y)$$

$$\int_0^h (R_3^y f)(x,y)dy = \frac{h^5}{720} f^{(0,4)}(x,\nu_2)$$

$\mu_1, \mu_2, \nu_1, \nu_2 \in [0,h]$, it follows that $ord(Q_1^x) = ord(Q_1^y) = 3$ and $ord(Q_3^x) = ord(Q_3^y) = 5$. Hence, $ord(Q_3^x) = ord(Q_3^y) = ord(Q_1^x) + ord(Q_1^y) - 1$.

Example 3. Let T_h be the standard triangle, $T_h = \{(x,y) \in \mathbb{R}^2 | \ x \geq 0, \ y \geq 0, \ x+y \leq h\}$, $f : T_h \to \mathbb{R}$ and $\Lambda(f) = \left\{ \left(0, \frac{h}{2}\right), f\left(\frac{h}{2}, 0\right), f\left(\frac{h}{2}, \frac{h}{2}\right) \right\}$. Let P be the operator that interpolate the information $\Lambda(f)$, i.e.

$$(Pf)(x,y) = \frac{h-2x}{h} f\left(0, \frac{h}{2}\right) + \frac{h-2y}{h} f\left(\frac{h}{2}, 0\right) + \frac{2x+2y-h}{h} f\left(\frac{h}{2}, \frac{h}{2}\right)$$

and

$$f = Pf + Rf \tag{14}$$

the interpolation formula suitable to P.

Let

$$\int\int_{T_h} f(x,y)dxdy = Q(f) + R(f) \tag{15}$$

be the cubature formula generated by (14), i.e.

$$Q(f) = \frac{h^2}{6} \left[f\left(0, \frac{h}{2}\right) + f\left(\frac{h}{2}, 0\right) + f\left(\frac{h}{2}, \frac{h}{2}\right) \right]$$

and

$$R(f) = \int\int_{T_h} (Rf)(x,y)dxdy.$$

Remark 7. $dex(Q) = 2$, although $dex(P) = 1$.

Theorem 8. *Formula (15) is a homogeneous cubature formula of interpolations type.*

Proof. Suppose that $f \in B_{12}(0,0)$ on T_h. By Peano's theorem ($dex(Q) = 2$), we have

$$R(f) = \int_0^h K_{30}(x,y,s) f^{(3,0)}(s,0)ds + \int_0^h K_{21}(x,y,s) f^{(2,1)}(s,0)ds \tag{16}$$

$$+ \int_0^h K_{03}(x,y,t) f^{(0,3)}(0,t)dt + \iint\limits_{T_h} K_{12}(x,y,s,t) f^{(1,2)}(s,t)dsdt$$

where

$$K_{30}(s) = \frac{(h-s)^4}{24} - \frac{h^2}{6}\left(\frac{h}{2} - s\right)_+^2$$

$$K_{21}(s) = \frac{(h-s)^4}{24} - \frac{h^3}{12}\left(\frac{h}{2} - s\right)_+$$

$$K_{03}(t) = K_{30}(t)$$

$$K_{12}(s,t) = \frac{(h-s-t)^3}{6} - \frac{h^2}{6}\left(\frac{h}{2} - s\right)_+^0 \left(\frac{h}{2} - t\right)_+$$

As $K_{30} \geq 0$, $K_{03} \geq 0$ on $[0,h]$ and

$$\int_0^h K_{30}(s)ds = \frac{1}{720}h^5, \quad \int_0^h K_{03}(t)dt = \frac{1}{720}h^5$$

respectively

$$\max_{0 \leq s \leq h} |K_{21}(s)| = \frac{1}{384}h^4, \quad \max_{T_h} |K(s,t)| = \frac{1}{12}h^3,$$

the proof follows from (16).

Example 4. An interesting homogeneous formula, for $f : T_h \to \mathbb{R}$, is obtained from the interpolation formula

$$f = Pf + Rf$$

when

$$(Pf)(x,y) = f\left(\frac{h}{3}, \frac{h}{3}\right)$$

namely

$$\iint\limits_{T_h} f(x,y)dxdy = Q(f) + R(f) \tag{17}$$

with

$$Q(f) = \frac{h^2}{2} f\left(\frac{h}{3}, \frac{h}{3}\right).$$

It is easy to verify that $dex(Q) = 1$, while $dex(P) = 0$.

Theorem 9. *Formula (17) is a homogeneous cubature formula.*

Proof. If $f \in B_{11}(0,0)$ on T_h, by Peano's theorem one obtain

$$|R(f)| \leq \frac{h^4}{72}\left[\|f^{(2,0)}(\cdot,0)\| + \|f^{(0,2)}(0,\cdot)\| + \frac{89}{27}\|f^{(1,1)}\|\right].$$

3. Methods for nonlinear equations on $\mathbb{R}$

For $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$, one considers the equation

$$f(x) = 0, \quad x \in \Omega. \tag{18}$$

Let $F : D^n \to D$, $D \subset \Omega$, be an iterative method for solutions of the equation (18), i.e. for given $(x_0, \ldots, x_{n-1}) \in D^n$, F generates the sequence

$$x_0, x_1, \ldots, x_{n-1}, x_n, \ldots \tag{19}$$

where

$$x_i = F(x_{i-n}, \ldots, x_{i-1}), \quad i = n, \ldots$$

If the sequence (19) converges to a solution, say x^*, of the equation (18), F is a convergent method.

The number p with the property that

$$\lim_{x_i \to x^*} \frac{x^* - F(x_{i-n+1}, \ldots, x_i)}{(x^* - x_i)^p} = c, \quad c \in \mathbb{R} \setminus \{0\},$$

is the order of F ($ord(F) = p$).

An efficient procedure to construct numerical methods for the solution of the equation (18), is based on inverse interpolation. Namely, if g is the inverse of f, $g = f^{-1}$, and $x^* \in D$ is a solution of the equation (18), $f(x^*) = 0$, then $x^* = g(0)$. Inverse interpolation procedure means that the inverse function g is approximated by an interpolation operator P and $x^* \approx (Pg)(0)$.

For example [7], using Taylor interpolation is obtained the one step method

$$F_n^T(x_i) = x_i - \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} (f(x_i))^k g^{(k)}(f(x_i))$$

with $ord(F_n^T) = n$.

Also, using Lagrange interpolation is obtained the multistep method

$$F_n^L(x_{i-n}, \dots, x_i) = \sum_{k=0}^{n} \frac{f_{i-n} \cdots f_{i-n+k-1} f_{i-n+k+1} \cdots f_i}{(f_{i-n} - f_k) \dots (f_{i-n+k-1} - f_k)(f_{i-n+k+1} - f_k) \dots (f_i - f_k)} x_k$$

with $ord(F_n^L) = \rho$, the positive solution of the equation

$$t^{n+1} - t^n - \cdots - t - 1 = 0$$

i.e., $1 < \rho < 2$.

An interesting class of methods is given by Abel-Goncharov interpolation operator P, defined by

$$(Pf)(x) = \sum_{k=0}^{n} p_k(x) f^{(k)}(x_k)$$

where

$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x - x_0 \\ p_k(x) &= \frac{1}{k!} \left[x^k - \sum_{j=0}^{k-1} p_j(x) x_j^{k-j} \right], \ k = 2, \dots, n. \end{aligned}$$

Applying the operator P to the function $g = f^{-1}$, for the interpolation nodes $x_{i-n}, \dots, x_i$, one obtains

$$F_n^{AG}(x_{i-n}, \dots, x_i) = \sum_{k=i-n}^{i} p_{n-i-k}(0) g^{(n-i-k)}(f(x_k))$$

For example,

$$F_1^{AG}(x_{i-1}, x_i) = x_{i-1} - \frac{f(x_{i-1})}{f'(x_i)},$$

is a new modified of Newton-Raphson method.

References

[1] Barnhill, R.E., Dube, R.P., Little, F.F., *Properties of Shepard's surfaces*, Rocky Mountain J. of Math., **13**(2), 1983, 365-382.

[2] Cătinaş, T., *The combined Shepard-Abel-Goncharov univariate operator*, Rev. Anal. Numer. Theor. Approx., 32, no.1, 2003, 11-20.

[3] Coman, Gh., *Numerical Analysis*, Ed. Libris, Cluj-Napoca, 1995.

[4] Coman, Gh., *Multivariate approximation schemes and the approximation of linear functionals*, Mathematica, 16(39), 1974, 229-249.

[5] Coman, Gh., *Homogeneous cubature formulas*, Studia Univ. Babeş-Bolyai, Mathematica, 38, no.2, 1993, 91-101.

[6] Coman, Gh., *Homogeneous multivariate approximation formulas with applications in numerical integration*, Research Seminars, Babeş-Bolyai University, Faculty of Mathematics, Preprint no.4, 1985, 46-63.

[7] Coman, Gh. et al., *Interpolation operators*, Ed. Casa Cărţii de Ştiinţă, Cluj-Napoca, 2004.

[8] Gordon, W. J., *Distributive lattices and approximation of multivariate functions*, Proc. Symp. Approximation with Special Emphasis on Spline Functions, Madison, Wisconsin (Ed. I. J. Schoenberg), 1969, 158-177.

[9] Sard, A., *Linear approximation*, AMS, Providence, RI, 1963.

[10] Schoenberg, I. J., *On best approximation of linear operators*, Indag. Math. 26, 1964, 155-163.

[11] Stancu, D. D., *The remainder of certain linear approximation formulas in two variables*, SIAM J. Numer. Anal., Ser. B, 1, 1964, 137-163.

[12] Traub, J. F., *Iterative methods for the solution of equations*, Prentice Hall, Inc., Englewood Cliffs, 1964.

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