STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 4, December 2004

SPECTRAL RADIUS OF QUOTIENT BOUNDED OPERATOR

SORIN MIREL STOIAN

Abstract. We introduce the spectral radius $r_{\mathcal{P}}(T)$ for a quotient bounded operator on a locally convex space X. Similarly to the case of bounded operator on a Banach space we prove that the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda, T)$, whenever $|\lambda| > r_{\mathcal{P}}(T)$, and $|\sigma(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T)$.

1. Introduction

The spectral theory for a linear operator on Banach space X is well developed and we have useful tools for use this theory. For example, the spectral radius of such operator T is defined by the Gelfand formula $r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$ and $|\sigma(Q, T)| = r(T)$.

Further it is known that the rezolvent $R(\lambda, T)$ is given by the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$, whenever $|\lambda| > r(T)$.

If we want to generalize this theory on locally convex space X one major difficulty is that is not clear which class of operators we can use, because there are several non-equivalent ways of defining bounded operators on X. The concept of bounded element of a locally convex algebra was introduced by Allan [1]. An element is said to be bounded if some scalar multiple of it generates a bounded semigroup.

Definition 1.1. Let X be a locally convex algebra. The radius of boundness of an element $x \in X$ is the number

$$\beta(x) = \inf\{\alpha > 0 | \text{ the set } \{(\alpha x)^n\}_{n\geq 1} \text{ is bounded}\}.$$

Received by the editors: 01.06.2004.
2000 *Mathematics Subject Classification.* 47S10.
Key words and phrases. Spectral radius, locally convex space, Neumann series.

In this paper we consider the class of quotient bounded operators, which was introduced in Appendix A by A. Michael [8], and later was studied by T. Moore [9] and A. Chilana [2].

Throught this paper all locally convex spaces will be assumed Hausdorff, over complex field $\mathbb{C}$, and all operators will be linear. If X and Y are topological vector spaces we denote by $L(X,Y)$ ($\mathcal{L}(X,Y)$) the algebra of linear operators (continuous operators) from X to Y.

Any family $\mathcal{P}$ of seminorms who generate the topology of locally convex space X (in the sense that the topology of X is the coarsest with respect to which all seminorms of $\mathcal{P}$ are continuous) will be called a calibration on X. A calibration is characterized by the property, that for every seminorms $p \in \mathcal{P}$ and every constant $\varepsilon > 0$ the sets

$$S(p,\varepsilon) = \{x \in X \mid p(x) < \varepsilon\},$$

constitute a neighbourhoods sub-base at 0. A calibration on X will be principal if it is directed. The set of calibration for X is denoted by $\mathcal{C}(X)$.

Any family of seminorms on a linear space is partially ordered by relation "$\leq$", where

$$p \leq q \;\Leftrightarrow\; p(x) \leq q(x),\; \forall\, x \in X.$$

A family of seminorms is preordered by relation "$\prec$", where

$$p \prec q \;\Leftrightarrow\; \text{there exists some } r > 0 \text{ such that } p(x) \leq rq(x),\; \forall\, x \in X.$$

If $p \prec q$ and $q \prec p$, we write $p \approx q$.

Definition 1.2. Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called Q-equivalent (denoted $\mathcal{P}_1 \approx \mathcal{P}_2$) provided:

a) for each $p_1 \in \mathcal{P}_1$ there exists $p_2 \in \mathcal{P}_2$ such that $p_1 \approx p_2$;

b) for each $p_2 \in \mathcal{P}_2$ there exists $p_1 \in \mathcal{P}_1$ such that $p_2 \approx p_1$.

It is obvious that two Q-equivalent and separating families of seminorms on a linear space generate the same locally convex topology.

Similar to the norm of an operator on a normed space we define the mixed operator seminorm of an operator between locally convex spaces. If $(X, \mathcal{P})$, $(Y, \mathcal{Q})$ are locally convex spaces, then for each $p, q \in \mathcal{P}$ the application $m_{pq} : L(X, Y) \to \mathbb{R} \cup \{\infty\}$, defined by

$$m_{pq}(T) = \sup_{p(x) \neq 0} \frac{q(Tx)}{p(x)},$$

is called the mixed operator seminorm of T associated with p and q. When $X = Y$ and $p = q$ we use notation $\widehat{p} = m_{pp}$.

Lemma 1.3. (V. Troistky [10]) *If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces and $T \in L(X, Y)$, then*

1) $m_{pq}(T) = \sup_{p(x)=1} q(Tx) = \sup_{p(x)\leq 1} q(Tx),\ \forall\, p \in \mathcal{P},\ \forall\, q \in \mathcal{Q}$;
2) $q(Tx) \leq m_{pq}(T)p(x),\ \forall\, x \in X$, whenever $m_{pq}(T) < \infty$.

Corollary 1.4. *If $(X, \mathcal{P}), (Y, \mathcal{Q})$ are locally convex spaces and $T \in L(X, Y)$, then*

$$m_{pq}(T) = \inf\{M > 0 |\ q(Tx) \leq Mp(x),\ \forall\, x \in X\},$$

whenever $m_{pq}(T) < \infty$.

Proof. If $p, q \in \mathcal{P}$ then from previous lemma we have

$$q(Tx) \leq m_{pq}(T)p(x),\ \forall\, x \in X.$$

If $M > 0$ such that

$$q(Tx) \leq Mp(x),\ \forall\, x \in X,$$

then using lemma 1.3.(1) we obtain

$$m_{pq}(T) = \sup_{p(x)=1} q(Tx) \leq M.$$

Definition 1.5. An operator T on a locally convex space X is quotient bounded with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ if for every seminorm $p \in \mathcal{P}$ there exists some $c_p > 0$ such that

$$p(Tx) \leq c_p p(x),\ \forall\, x \in X.$$

The class of all quotient bounded operators with respect to a calibration $\mathcal{P} \in \mathcal{C}(X)$ is denoted by $Q_{\mathcal{P}}(X)$.

Lemma 1.6. *If X is a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$, then for every $p \in \mathcal{P}$ the application $\widehat{p} : Q_{\mathcal{P}}(X) \to \mathbb{R}$ defined by*

$$\widehat{p}(T) = \{r > 0 |\ p(Tx) \leq rp(x),\ \forall\ x \in X\},$$

is a submultiplicative seminorm on $Q_{\mathcal{P}}(X)$, satisfying $\widehat{p}(I) = 1$.

We denote by $\widehat{\mathcal{P}}$ the family $\{\widehat{p} | p \in \mathcal{P}\}$.

Proposition 1.7. (G. Joseph [7]) *Let X be a locally convex space and $\mathcal{P} \in \mathcal{C}(X)$.*

1) $Q_{\mathcal{P}}(X)$ is a unital subalgebra of the algebra of continuous linear operators on X;

2) $Q_{\mathcal{P}}(X)$ is a unital locally multiplicative convex algebra (l.m.c.-algebra) with respect to the topology determined by $\widehat{\mathcal{P}}$;

3) If $\mathcal{P}' \in \mathcal{C}(X)$ such that $\mathcal{P} \approx \mathcal{P}'$, then $Q_{\mathcal{P}'}(X) = Q_{\mathcal{P}}(X)$ and $\widehat{\mathcal{P}} = \widehat{\mathcal{P}}'$;

4) The topology generated by $\widehat{\mathcal{P}}$ on $Q_{\mathcal{P}}(X)$ is finer than the topology of uniform convergence on bounded subsets of X.

Lemma 1.8. *If X is a sequentially complete convex space, then $Q_{\mathcal{P}}(X)$ is a sequentially complete m-convex algebra for all $\mathcal{P} \in \mathcal{C}(X)$.*

Proof. Let $\mathcal{P} \in \mathcal{C}(X)$ and $(T_n)_n \subset Q_{\mathcal{P}}(X)$ be a Cauchy sequence. Then, for each $\varepsilon > 0$ and each $\widehat{p} \in \widehat{\mathcal{P}}$ there exists some index $n_{p,\varepsilon} \in \mathbb{N}$ such that

$$|\widehat{p}(T_n) - \widehat{p}(T_m)| \leq \widehat{p}(T_n - T_m) < \varepsilon,\ \forall\ n, m \geq n_{p,\varepsilon}. \tag{1}$$

From the previous relation it follows that $(\widehat{p}(T_n))_n$ is convergent sequence of real numbers, for each $\widehat{p} \in \widehat{\mathcal{P}}$. If $x \in X$, then

$$p(T_n x - T_m x) \leq \widehat{p}(T_n - T_m)p(x),\ \forall\ p \in \mathcal{P}, \tag{2}$$

so $(T_n(x))_n \subset X$ is a Cauchy sequence. But, since X is sequentially complete and Hausdorff, there exists an unique element $y \in X$ such that

$$\lim_{n\to\infty} T_n x = y.$$

Therefore, the operator $T : X \to X$ defined by

$$T(x) = \lim_{n\to\infty} T_n x,\ \forall\ x \in X,$$

is well defined. It is obvious that T is linear operator. Using the continuity of seminorms $\widehat{p} \in \widehat{\mathcal{P}}$ we have

$$p(Tx) = p\left(\lim_{n\to\infty} T_n x\right) = \lim_{n\to\infty} p(T_n x) \leq \lim_{n\to\infty} \widehat{p}(T_n)p(x) = c_p p(x),$$

for all $x \in X$ and for each $p \in \mathcal{P}$ (where $c_p = \lim_{n\to\infty} \widehat{p}(T_n)$).

This implies that $T \in Q_{\mathcal{P}}(X)$. Now we prove that $T_n \to T$ in $Q_{\mathcal{P}}(X)$.

From relations (1) and (2) it follows that for each $\varepsilon > 0$ and each $\widehat{p} \in \widehat{\mathcal{P}}$ there exists $n_{p,\varepsilon} \in \mathbb{N}$ such that

$$p(T_n x - T_m x) < \varepsilon p(x),\ \forall\ n, m \geq n_{p,\varepsilon}$$

so

$$p(T_n x - Tx_) \leq \varepsilon p(x),\ \forall\ n \geq n_{p,\varepsilon}.$$

This implies that

$$\widehat{p}(T_n - T) \leq \varepsilon,\ \forall\ n \geq n_{p,\varepsilon},$$

which prove that $T_n \to T$ in $Q_{\mathcal{P}}(X)$ and $Q_{\mathcal{P}}(X)$ is a sequentially complete m-convex algebra. $\square$

Given $(X, \mathcal{P})$, for each $p \in \mathcal{P}$ let N^p denote the null space $\{x|\ p(x) = 0\}$ and X_p the quotient space X/N^p. For each $p \in \mathcal{P}$ consider the natural mapping

$$x \to x_p \equiv x + N^p \text{ (from } X \text{ to } X_p).$$

It is obvious that X_p is normed space, for each $p \in \mathcal{P}$, with norm defined by $\|x_p\|_p = p(x)$. Consider the algebra homomorphism $T \to T^p$ of $Q_{\mathcal{P}}(X)$ into $\mathcal{L}(X_p)$ defined by

$$T^p(x_p) = (Tx)_p,\ \forall\ x \in X.$$

This operator are well defined because $T(N^p) \subset N^p$. Moreover, for each $p \in \mathcal{P}$, $\mathcal{L}(X_p)$ is a unital normed algebra and we have

$$\|T_p\|_p = \sup\{\|T_p x_p\|_p | \ \|x_p\|_p \leq 1 \text{ for } x_p \in X_p\} =$$

$$= \sup\{p(Tx) | \ p(x) \leq 1 \text{ for } x \in X\}.$$

For $p \in \mathcal{P}$ consider the normed space $(\widetilde{X}_p, \|\cdot\|_p)$ the completion of $(X_p, \|\cdot\|_p)$. If $T \in Q_{\mathcal{P}}(X)$, then the operator T^p has a unique continuous linear extension $\widetilde{T}^p$ on $(\widetilde{X}, \|\cdot\|_p)$.

Definition 1.9. Let $(X, \mathcal{P})$ be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We say that $\lambda \in \rho(Q_{\mathcal{P}}, T)$ if the inverse of $\lambda I - T$ exists and $(\lambda I - T)^{-1} \in Q_{\mathcal{P}}(X)$.

Spectral sets $\sigma(Q_{\mathcal{P}} T)$ are defined to be complements of rezolvent sets $\rho(Q_{\mathcal{P}}, T)$.

For each $p \in \mathcal{P}$ we denote by $\sigma(X_p, T^p)$ $(\sigma(\widetilde{X}_p, \widetilde{T}^p))$ the spectral set of the operator T^p in $\mathcal{L}(X_p)$ (respectively the rezolvent set of $\widetilde{T}^p$ in $\mathcal{L}(\widetilde{X}_p)$). The rezolvent set of the operator T^p in $\mathcal{L}(X_p)$ (respectively the spectral set of $\widetilde{T}^p$ in $\mathcal{L}(\widetilde{X}_p)$) is denoted by $\rho(X_p, T^p)$ $(\rho(\widetilde{X}_p, \widetilde{T}^p))$.

Lemma 1.10. (J. R. Gilles, G. Joseph, B. Sims [6]) *Let $(X, \mathcal{P})$ be a sequentially complete convex space and $T \in Q_{\mathcal{P}}(X)$. Then T is invertible in $Q_{\mathcal{P}}(X)$ if and only if $\widetilde{T}^p$ is invertible in $\mathcal{L}(\widetilde{X}_p)$ for all $p \in \mathcal{P}$.*

Corollary 1.11. (J. R. Gilles, G. Joseph, B. Sims [6]) *If $(X, \mathcal{P})$ is a sequentially complete convex space and $T \in Q_{\mathcal{P}}(X)$, then*

$$\sigma(Q_{\mathcal{P}}, T) = \cup\{\sigma(X_p, T^p) | \ p \in \mathcal{P}\} = \cup\{\sigma(\widetilde{X}_p, \widetilde{T}^p) | \ p \in \mathcal{P}\}.$$

2. **Spectral radius of quotient bounded operators**

Let $(X, \mathcal{P})$ be a locally convex space and $T \in Q_{\mathcal{P}}(X)$. We said that T is bounded element of the algebra $Q_{\mathcal{P}}(X)$ if it is a bounded element of $Q_{\mathcal{P}}(X)$ in the sense of G. R. Allan [1]. The class of bounded elements of $Q_{\mathcal{P}}(X)$ is denoted by $(Q_{\mathcal{P}}(X))_0$.

Definition 2.1. If $(X, \mathcal{P})$ is a locally convex space and $T \in Q_{\mathcal{P}}(X)$ we denote by $r_{\mathcal{P}}(T)$ the radius of boundness of operator T in $Q_{\mathcal{P}}(X)$, i.e.

$$r_{\mathcal{P}}(T) = \inf\{\alpha > 0 |\ \alpha^{-1}T \text{ generates a bounded semigroup in } Q_{\mathcal{P}}(X)\}.$$

We said that $r_{\mathcal{P}}(T)$ is the $\mathcal{P}$-spectral radius of the operator T.

Proposition 1.7(3) implies that for each $\mathcal{P}' \in \mathcal{C}(X)$, $\mathcal{P} \approx \mathcal{P}'$, we have $Q_{\mathcal{P}'}(X) = Q_{\mathcal{P}}(X)$, so if $\mathcal{H}$ is a Q-equivalence class in $\mathcal{C}(X)$, then

$$r_{\mathcal{P}}(T) = r_{\mathcal{P}'}(T),\ \forall\ \mathcal{P}, \mathcal{P}' \in \mathcal{H}.$$

Since $Q_{\mathcal{P}}(X)$ is a m-convex algebra, for each $\mathcal{P} \in \mathcal{C}(X)$, the propositions 2.2-2.5 follows from the results proved by G. A. Allan [1] and I. Colojoara [3].

Proposition 2.2. *If X is a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have:*

1) $r_{\mathcal{P}}(T) \geq 0$ and

$$r_{\mathcal{P}}(\lambda T) = |\lambda| r_{\mathcal{P}}(T),\ \forall\ \lambda \in \mathbb{C},$$

where by convention $0\infty = \infty$;

2) $r_{\mathcal{P}}(T) < +\infty$ if and only if $T \in (Q_{\mathcal{P}}(X))_0$;

3) $r_{\mathcal{P}}(T) = \inf\left\{\lambda > 0 |\ \lim\limits_{n\to\infty} \frac{T^n}{\lambda^n} = 0\right\}$.

Proposition 2.3. *If X is a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$, then for each $T \in Q_{\mathcal{P}}(X)$ we have:*

$$r_{\mathcal{P}}(T) = \sup\left\{\limsup_{n\to\infty} (\widehat{p}(T^n))^{1/n} |\ p \in \mathcal{P}\right\} =$$

$$= \sup\left\{\lim_{n\to\infty} (\widehat{p}(T^n))^{1/n} |\ p \in \mathcal{P}\right\} = \sup\left\{\inf_{n\geq 1} (\widehat{p}(T^n))^{1/n} |\ p \in \mathcal{P}\right\}.$$

Proposition 2.4. *Let X be a locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$.*

1) If $T \in (Q_{\mathcal{P}}(X))$, then

$$\lim_{n\to\infty} \frac{T^n}{\lambda^n} = 0,\ \forall\ |\lambda| > r_{\mathcal{P}}(T);$$

2) If $T \in (Q_{\mathcal{P}}(X))_0$ *and* $0 < |\lambda| < r_{\mathcal{P}}(T)$*, then the set* $\left\{\frac{T^n}{\lambda^n}\right\}_{n\geq 1}$ *is unbounded;*

3) For each $T \in Q_{\mathcal{P}}(X)$ *and every* $n > 0$ *we have*

$$r_{\mathcal{P}}(T^n) = r_{\mathcal{P}}(T)^n.$$

Proposition 2.5. *Let* X *be a locally convex algebra and* $\mathcal{P} \in \mathcal{C}(X)$*. Then:*

1) $r_{\mathcal{P}}(T+S) \leq r_{\mathcal{P}}(T) + r_{\mathcal{P}}(S),\ \forall\ T,S \in Q_{\mathcal{P}}(X)$ *which have property* $TS = ST$*;*

2) $r_{\mathcal{P}}(TS) \leq r_{\mathcal{P}}(T) r_{\mathcal{P}}(S),\ \forall\ T,S \in Q_{\mathcal{P}}(X)$ *which have property* $TS = ST$*.*

From real analysis we have the following lemma.

Lemma 2.6. (V. Troistky [10]) *If* $(t_n)_n$ *is a sequence in* $\mathbb{R}^* \cup \{\infty\}$ *then*

$$\limsup_{n\to\infty} \sqrt[n]{t_n} = \inf\left\{v > 0 |\ \lim_{n\to\infty} \frac{t_n}{v^n} = 0\right\}.$$

This lemma implies that for a bounded operator on Banach space we have

$$r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|} = \inf\Big\{v > 0 |\ \text{sequence } \left(\frac{T^n}{v^n}\right)_n \text{ converge to zero}$$

$$\text{in operator norm topology}\Big\}.$$

If we consider this relation as an alternative definition of the spectral radius, then proposition 2.2(3) implies that $\mathcal{P}$-spectral radius of an quotient bounded operator can be considered to be natural generalization of the spectral radius of bounded operator on Banach space.

Proposition 2.7. *Let* X *be a sequentially complete locally convex algebra and* $\mathcal{P} \in \mathcal{C}(X)$*. If* $T \in (Q_{\mathcal{P}}(X))_0$ *and* $|\lambda| > r_{\mathcal{P}}(T)$*, then the Neumann series* $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ *converges to* $R(\lambda, T)$ *(in* $Q_{\mathcal{P}}(X)$*) and* $R(\lambda,T) \in Q_{\mathcal{P}}(X)$*.*

Proof. If $|\lambda| > r_{\mathcal{P}}(T)$, then there exists $\beta \in \mathbb{C}$ such that $0 < |\beta| < 1$ and $r_{\mathcal{P}}(T) < \beta\lambda$. From proposition 2.4(1) we obtain that for each $\varepsilon > 0$ and every $p \in \mathcal{P}$, there exists some index $n_{p,\varepsilon} \in \mathbb{N}$, with property

$$\widehat{p}\left(\frac{T^n}{(\beta\lambda)^n}\right) < \varepsilon,\ \forall\ n \geq n_{p,\varepsilon}.$$

Therefore, using corollary 1.4 we obtain

$$p\left(\frac{T^n}{(\beta\lambda)^n}x\right) \leq \widehat{p}\left(\frac{T^n}{(\beta\lambda)^n}\right)p(x) < \varepsilon p(x),\ \forall\ n \geq n_{p,\varepsilon},\ \forall\ x \in X.$$

Since $0 < |\beta| < 1$, there exists $n_0 \in \mathbb{N}$, such that

$$\sum_{k=n}^{m} |\beta|^k < 1,\ \forall\ m > n \geq n_0.$$

From a previous relation result that for each $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists an index $m_{p,\varepsilon} = \max\{n_{p,\varepsilon}, n_0\} \in \mathbb{N}$, for which we have

$$p\left(\sum_{k=n}^{m} \frac{T^k}{\lambda^k}x\right) \leq \varepsilon\left(\sum_{k=n}^{m} |\beta|^k\right)p(x) < \varepsilon p(x), \tag{3}$$

for every $m > n \geq m_{p,\varepsilon}$ and every $x \in X$.

Therefore, for each $x \in X$, $\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}x\right)_{m\geq 0}$ is a Cauchy sequence.

But X is sequentially complete, so for every $x \in X$ there exists an unique element $y \in X$ such that

$$y = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}x.$$

We consider the operator $S : X \to X$ given by

$$S(x) = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}x,\ \forall\ x \in X.$$

It is obvious that S is linear operator. Moreover, from equality

$$\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}(\lambda x - Tx) = x - \frac{T^{m+1}}{\lambda^{m+1}}x,\ \forall\ x \in X,$$

result that if $m \to \infty$ then

$$S(\lambda x - Tx) = x,\ \forall\ x \in X.$$

Hence $S(\lambda I - T) = I$, we prove $(\lambda I - T)S = I$. From continuity of the operator T result that

$$STx = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} Tx = \lim_{m\to\infty} T\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x\right) =$$

$$= T\left(\lim_{m\to\infty} \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x\right) = TSx,$$

for all $x \in X$, therefore

$$S(\lambda I - T) = (\lambda I - T)S = I.$$

The definition of $\mathcal{P}$-spectral radius implies that family $\left(\frac{T^n}{(\beta\lambda)^n}\right)_n$ is bounded in $Q_{\mathcal{P}}(X)$, therefore for every $p \in \mathcal{P}$ there exists a constant $\varepsilon_p > 0$ with property

$$\widehat{p}\left(\frac{T^n}{(\beta\lambda)^n}\right) < \varepsilon_p, \ \forall\, n \geq 1.$$

Using again corollary 1.4 we have

$$p\left(\frac{T^n}{\lambda^n} x\right) < \varepsilon_p |\beta|^n p(x), \ \forall\, n \geq 1, \ \forall\, x \in X.$$

Therefore, for every $p \in \mathcal{P}$ there exists some $\varepsilon_p > 0$ with property

$$p\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x\right) < \frac{\varepsilon_p}{|\lambda|}\left(\sum_{k=0}^{m} |\beta|^k\right) p(x) < \frac{\varepsilon_p}{|\lambda|} \frac{1}{1-|\beta|} p(x),$$

for every $m \geq 1$ and every $x \in X$, which implies that $S = R(\lambda, T) \in Q_{\mathcal{P}}(X)$.

If we write relation (3) under the form

$$p\left(\sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}} x - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} x\right) < \frac{\varepsilon}{|\lambda|} p(x),$$

then for $m \to \infty$ result that for every $\varepsilon > 0$ and every $p \in \mathcal{P}$ there exists some index $n_{p,\varepsilon} \in \mathbb{N}$, such that

$$p\left(Sx - \sum_{k=0}^{n} \frac{T^k}{\lambda^{k+1}} x\right) \leq \frac{\varepsilon}{|\lambda|} p(x), \ \forall\, n \geq n_{p,\varepsilon}, \ \forall\, x \in X.$$

This implies that the Neumann series $\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges to $R(\lambda, T)$ in $Q_{\mathcal{P}}(X)$. $\square$

Proposition 2.8. *Let X be a sequentially complete locally convex algebra and $\mathcal{P} \in \mathcal{C}(X)$. If $T \in Q_{\mathcal{P}}(X)$, then*

$$|\sigma(Q_{\mathcal{P}}, T)| = r_{\mathcal{P}}(T).$$

Proof. Inequality $|\sigma(Q_{\mathcal{P}}, T)| \le r_{\mathcal{P}}(T)$ is implied by previous proposition.

We prove now the reverse inequality. From corollary 1.11 we have

$$\sigma(Q_{\mathcal{P}}, T) = \cup\{\sigma(X_p, T^p) |\ p \in \mathcal{P}\} = \cup\{\sigma(\widetilde{X}_p, \widetilde{T}^p) |\ p \in \mathcal{P}\}.$$

So, if $|\lambda| > |\sigma(Q_{\mathcal{P}}, T)|$, then

$$|\lambda| > |\sigma(\widetilde{X}_p, \widetilde{T}^p)|,\ \forall\ p \in \mathcal{P}.$$

But, $\widetilde{X}_p$ is Banach space for each $p \in \mathcal{P}$, therefore

$$|\sigma(\widetilde{S}_p, \widetilde{T}^p)| = r(\widetilde{X}_p, \widetilde{T}^p)$$

where $r(\widetilde{X}_p, \widetilde{T}^p)$ is spectral radius of bounded operator $\widetilde{T}^p$ in $\widetilde{X}_p$.

This observation implies that for each $p \in \mathcal{P}$ we have $\dfrac{T^{p^n}}{\lambda^n} \to 0$ in $\mathcal{L}(\widetilde{X}_p)$. This means that for any $\varepsilon > 0$ we must have $\|T^n\|_p \le (\varepsilon + |\sigma(Q_{\mathcal{P}}, T)|)^n$ for large n. Hence, by proposition 2.3 we have $r_{\mathcal{P}}(T) \le |\sigma(Q_{\mathcal{P}}, T)|$.

References

[1] G. R. Allan, *A spectral theory for locally convex algebras*, Proc. London Math. Soc. 15(1965), 399-421.

[2] A. Chilana, *Invariant subspaces for linear operators on locally convex spaces*, J. London Math. Soc., (2)2, 493-503.

[3] I. Colojoara, *Elemente de teorie spectrală*, Ed. Academiei R.S.R., Bucureşti, 1968.

[4] H. R. Dowson, *Spectral theory of linear operators*, Academic Press, 1978.

[5] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, Inc., 1965.

[6] J. R. Gilles, G. A. Joseph, D. O. Koehler, B. Sims, *On numerical ranges of operators on locally convex spaces*, J. Austral. Math. Soc. 20 (Series A) (1975), 468-482.

[7] G. A. Joseph, *Boundness and completeness in locally convex spaces and algebras*, J. Austral. Math. Soc. 24 (Series A) (1977), 50-63.

[8] A. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc., 11, 1952.

[9] R. T. Moore, *Banach algebras of operators on locally convex spaces*, Bull. Am. Math. Soc., 74(1969), 69-73.

[10] V. G. Troistky, *Invariant subspace problem and spectral properties of bounded operators on Banach spaces, Banach Lattices and topological spaces*, Ph. D. Thesis, University of Illinois at Urbana-Champaign, 1999.

UNIVERSITY OF PETROŞANI, ROMANIA