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HÖLDER ESTIMATES OF HIGHER ORDER DERIVATIVES FOR EVOLUTIONARY MONGE-AMPÉRE EQUATION ON A RIEMANNIAN MANIFOLD

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Abstract. Let (V, g) be a compact Riemannian manifold. For $u \in C^2(V)$ we consider the form $g_{ij} + \nabla_{ij}u$. If the form is positive definite, it gives a new metric on V. The Monge-Ampére operator on V is the quotient of determinants: $M(u) = |g_{ij} + \nabla_{ij}u|/|g_{ij}|$. The paper deals with the Cauchy problem for the evolutionary Monge-Ampére type equation:

$$-\frac{\partial u}{\partial t} + \ln M(u) = f(t, x, u), \quad (t, x) \in [0, T] \times V,$$
$$u(0, x) = u_0(x).$$

Hölder estimates for higher order derivatives u_t and $\nabla_{ij}u$ of a solution u are proved.

1. Introduction

The paper deals with the apriory estimates of solutions of the Cauchy problem for the evolutionary Monge-Ampére type equation on Riemannian manifolds and continues [1],[2].

Let (V, g) be a smooth compact Riemannian manifold, $\dim V = m$. We consider the Levi-Civita connection on V, it defines the covariant differentiation on V. The Levi-Civita connection is the unique symmetric connection with vanishing torsion tensor, for which the covariant derivative of the metric tensor is zero. Let $x^1, \ldots, x^m$ be a local coordinate system on V, and $\partial_1, \ldots, \partial_m$, where $\partial_k = \frac{\partial}{\partial x^k}$, be

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the corresponding basic vector fields. Suppose $u(x)$ is a function on V at least twice continuously differentiable. By $\nabla_i u = \nabla_{\partial_i} u$ we denote the covariant derivative, and $\nabla_{ij} u = \nabla_i(\nabla_j u)$ second order covariant derivative. Let $(g_{ij}(x))$ be the matrix of the metric g in a local coordinates. We consider the form g_u with matrix $g_{ij}(x)+\nabla_{ij}u(x)$ in the local coordinate system. A function $u(x)$ is said to be **admissible** provided the form g_u is positive definite. An admissible function gives a new metric on V.

By $|g_{ij}|$ and $|g^u_{ij}|$ we denote determinants of metrics. The quotient $M(u)(x) = |g^u_{ij}(x)|/|g_{ij}(x)|$ is a positive function on V. We call $u \to M(u)$ **the Monge-Ampere type operator** by analogy with the classical Monge-Ampére operator. The distinction between $M(u)$ and the classical operator is the following. The classical operator is the Hesse matrix of a function u, but $M(u)$ contains sum of the matrix (g_{ij}) and Hesse matrix. The classical operator is defined on the convex set of symmetric positive definite matrix, for $M(u)$ we shall consider a bundle with convex fibres.

We consider the product $[0,T]\times V$ with the same metric g and connection ∇ for each $t \in [0,t]$. Let $u(t,x)$ be at least twice continuously differentiable function on $[0,T]\times V$ with respect to spatial variables. The function $u(t,x)$ is said to be *admissible* provided the form $g^u_{ij}(t) = g_{ij} + \nabla_{ij}u(t,\cdot)$ is positive definite for all $t \in [0,1]$. An admissible function $u(t,x)$ defines the family of metrics $g^u(t)$, $t \in [0,1]$ on V. Applying M to $u(t,x)$, we obtain the function $M(u)(t,x)$ depending on two variables.

We consider the evolutionary equation

$$-\frac{\partial u}{\partial t} + \ln M(u) = f(t,x,u), \qquad (t,x) \in [0,T]\times V, \tag{1}$$

with initial condition:

$$u(0,x) = u_0(x). \tag{2}$$

The stationary equation with $M(u)$ arises in some geometrical problem. For example, the condition that describes Einstein-Kähler manifolds is proportionality of the Ricci tensor and the metric tensor, it was first proposed by Einstein as the equation of the gravity field in vacuum. The question of existence of Einstein-Káhler metric leads to the stationary Monge-Ampére type equation. The proof of the famous Calabi conjecture, which asserts that every form representing the first Chern class is the Ricci form of some Kähler metric, proved in 1976 by S.T.Yau and T.Aubin independently, is based on the existence theorem for stationary Monge-Ampére equation (see [3], [4], [5],[6] for more details).

Evolutionary equations with classical Monge-Ampére operator on a bounded domain in the n-dimensional space arise in the problem of deformation of a hypersurface with rapid controlled by the mean curvature. Papers of many authors are devoted to the last problem, e.g. papers of N.Uraltseva, V.Oliker, N.Ivochkina, K.Tso, G.Huisken and others.

The aim of the paper is the Hölder constant estimates for the higher order derivatives for solutions of (1-2). In the proof we use the following estimates obtained in [1],[2].

Theorem 1. *([1], th.1) Let $u(t,x)$ be an admissible function and belong to $C\left([0,T],C^2(V)\right)$. By D denote the diameter of V. Then we have*

$$\max_{[0,t]\times V} |\nabla_x u| \leq 2D.$$

Theorem 2. *([1], th.2) Suppose $u(t,x)$ is an admissible solution of (1)-(2) and belongs $C\left([0,T],C^3(V)\right)\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Let the right hand side $f(t,x,u)$of equation (1) be bounded and have bounded first order partial derivatives, $f_u(t,x,u)\geq\delta>0$ on $[0,T]\times V\times R^1$. Then*

$$|u_t(x,t)|\leq M_1,$$

where M_1 depends on the diameter D, metric g, $\|u_0\|_{C^1(V)}$, $\|f\|_{C^1(V)}$, and δ.

As usual we denote by (g_{ij}) elements of matrix g in a local coordinates, (g^{ij}) elements of inverse matrix, also we denote by (g^u_{ij}) elements of matrix g_u and (g^{ij}_u) elements of corresponding inverse matrix.

Theorem 3. *Let $u(t,x)$ be an admissible solution of (1)-(2) and belong $C\left([0,T],C^4(V)\right)\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Suppose the right hand side $f(t,x,u)$ is bounded and has bounded partial derivatives up to the second order, $f_u(t,x,u)\geq\delta>0$ on $[0,T]\times V\times R^1$. Then all metrics generated by the solution $u(t,x)$ of (1)-(2) are uniformly equivalent, i.e. there are positive constants*

c_1, c_2, depending on the diameter D, metric tensor g, curvature tensor of V, $\|f\|_{C^2(V)}$, δ, $\|u_0\|_{C^2(V)}$, and independent on (t,x) such that

$$c_1 g_{ij}\xi^i\xi^j \le g^u_{ij}\xi^i\xi^j \le c_2 g_{ij}\xi^i\xi^j; \tag{3}$$

$$1/c_2 g^{ij}\xi_i\xi_j \le g^{ij}_u\xi_i\xi_j \le 1/c_1 g^{ij}\xi_i\xi_j. \tag{4}$$

for all $\xi = (\xi_1, \dots, \xi_m) \in R^m$.

Theorem 4. *Let $u(t,x)$ be a solution of (1-2) and belong to $C\left([0,T], C^4(V)\right) \bigcap C^1\left([0,T], C^2(V)\right) \bigcap C^2\left([0,T], C(V)\right)$. Under the assumptions of theorem 3 we have*

$$0 < m - \Delta u \le K,$$

where K depends on the same values as c_1, c_2 in theorem 3.

2. Some properties of the operator $M(u)$

Let us consider the set of all square matrix of order m, we identify it with R^{m^2}. Denote by $S \in R^{m^2}$ the subset of symmetric positive definite matrix. S is open and convex. Write $a = (a_{ij})$ for elements of S.

Let us cover V by finite number of local charts $(\Omega_k, \varphi_k)_{k=1}^q$ and choose open sets Ω'_k, $\bar{\Omega}'_k \subset \Omega_k$, such that $\varphi_k(\Omega'_k)$ convex in R^m and $\bigcup\limits_{k=1}^{q} \Omega'_k = V$. Fix an index k, we shall proceed throughout $\bar{\Omega}'_k$ in the local coordinates of chart (Ω_k, φ_k).

Fix $x \in \bar{\Omega}'_k$, then $g(x) \in S$. Denote by S_x the following subset in R^{m^2}:

$$S_x = \{a \in R^{m^2} \mid g(x) + a = (g_{ij}(x) + a_{ij}) \in S\}.$$

We consider the fibre bundle $\pi\colon \mathbf{S} \to \bar{\Omega}'_k$ with fibre $\pi^{-1}(x) = S_x$ and total space $\mathbf{S} = \bigcup\limits_{x\in\bar{\Omega}'_k} S_x$.

Fibres of the bundle π are open convex subset in R^{m^2} and every fibre is homeomorphic to S. The bundle π is trivializable, i.e. there is a homeomorphism $\varphi\colon \mathbf{S} \to \bar{\Omega}'_k \times S$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\varphi} & \bar{\Omega}'_k \times S \\ \pi \searrow & & \swarrow p_1 \\ & \bar{\Omega}'_k & \end{array},$$

where $p_1\colon \bar\Omega'_k \times S \to \bar\Omega'_k$ is the projection on the first factor, $p_1(x,a) = x$.

Indeed, we put $\varphi(x,a) = (x, a + g(x))$. The map φ is one-to-one:

1. if $(x_1, a_1 + g(x_1)) = (x_2, a_2 + g(x_2))$, then $x_1 = x_2$ and $a_1 + g(x_1) = a_2 + g(x_1) \quad\Rightarrow\quad a_1 = a_2$.
2. if $(x,b) \in \bar\Omega'_k \times S$, then $\varphi(x, b - g(x)) = (x, b - g(x) + g(x)) = (x,b)$.

The map φ is continuous due to continuity of g, the inverse map $\varphi^{-1}(x,b) = (x, b - g(x))$ is also continuous .

Together with $\pi\colon \mathbf{S} \to \bar\Omega'_k$ we consider the bundle $\bar\pi\colon [0,T]\times\mathbf{S} \to [0,T]\times\bar\Omega'_k$ whose fibre over (t,x) coincides with the fibre S_x of π over x: $\bar\pi^{-1}(t,x) = \pi^{-1}(x) = S_x$. By $\bar{\mathbf{S}} = [0,T]\times\mathbf{S}$ we denote the total space of the bundle $\bar\pi$.

On the bundle $\mathbf{S}$ we consider the following function $F\colon \mathbf{S} \to R^1$:

$$F(x,a) = \ln\frac{|g(x)+a|}{|g(x)|}.$$

We extend F identically to $\bar{\mathbf{S}}$: $F(t,x,a) = F(x,a)$. It is easily seen that the restriction F to a fibre S_x of the bundles $\mathbf{S}$ and $\bar{\mathbf{S}}$ is a convex function of m^2 variables. Indeed, $\frac{\partial F}{\partial(a_{ij})} = g_a^{ij}$, where g_a^{ij} is an element of the inverse matrix $(g(x)+a)^{-1}$. Then $\frac{\partial^2 F}{\partial(a_{ij})\partial(a_{kl})} = -g_a^{ik}g_a^{lj}$ ([1], lemma 1). Thus $\frac{\partial^2 F}{\partial a_{ij}\partial a_{kl}}\xi_{ij}\xi_{kl} = -g_a^{ik}g_a^{lj}\xi_{ij}\xi_{kl}$ is a negative definite form, i.e. the function $F|_{S_x}$ is a convex function of m^2 variables.

Let $\Lambda\colon [0,T]\times\bar\Omega'_k \to \bar{\mathbf{S}}$, $\Lambda(t,x) = (t,x,\lambda(t,x))$, $\lambda(t,x)\in S_x$, be a continuous section of the bundle $\bar\pi$. Assume that there are exist positive constants c_1, c_2 such that

$$c_1|\xi|^2 \le (g_{ij}(x) + \lambda_{ij}(t,x))\xi^i\xi^j \le c_2|\xi|^2 \tag{5}$$

for all $(t,x)\in[0,T]\times\bar\Omega'_k$.

We consider the superposition

$$F(x,\lambda(t,x)) = \ln\frac{|g(x)+\lambda(t,x)|}{|g(x)|}.$$

Denote by

$$\rho(z_1,z_2) = |t-\tau|^{1/2} + |x-y| \tag{6}$$

the parabolic distance between points $z_1 = (t,x), z_2 = (\tau,y) \in [0,T]\times\bar\Omega'_k$.

Lemma 1. *Suppose the metric g and section Λ_{ij} are $C^{2+\alpha}$. Let M^α_Λ be the maximal Hölder constant for λ_{ij}. Then*

$$|F(x,\lambda(z_1)) - F(y,\lambda(z_2))| \le N_1 \rho^\alpha(z_1,z_2) + N_2|x-y|^\alpha,$$

for all $z_1 = (t,x), z_2 = (\tau,y) \in [0,T]\times\bar{\Omega}'_k$ with constant N_1 depending on $c_1, m, \|g\|_{C^{2+\alpha}}, M^\alpha_\Lambda$ and constant N_2 depending on m, $\|g\|_{C^{2+\alpha}}$.

Proof.

$$\begin{aligned} F(x,\lambda(t,x)) - F(y,\lambda(\tau,y)) &= \ln\tfrac{|g(x)+\lambda(t,x)|}{|g(x)|} - \ln\tfrac{|g(y)+\lambda(\tau,y)|}{|g(y)|} = \\ &= [\ln|g(x)+\lambda(t,x)| - \ln|g(y)+\lambda(\tau,y)|] + [\ln|g(y)| - \ln|g(x)|]. \end{aligned} \tag{7}$$

We start with the first term. For all $\theta\in[0,1]$ the form $g_\theta = \theta[g(x)+\lambda(t,x)] + (1-\theta)[g(y)+\lambda(\tau,y)]$ is positive definite. Let us consider the function $\varphi(\theta) = \ln|g_\theta|$. We have

$$\varphi'(\theta) = \frac{1}{|g_\theta|} g^{ij}_\theta |g_\theta| \frac{d}{d\theta}(g_\theta)_{ij} = g^{ij}_\theta[g_{ij}(x) - g_{ij}(y) + \lambda_{ij}(t,x) - \lambda_{ij}(\tau,y)],$$

where $(g_\theta)_{ij}$ are elements of matrix g_θ, g^{ij}_θ are elements of the inverse matrix. Then

$$\begin{aligned} \ln|g(x)+\lambda(t,x)| - \ln|g(y)+\lambda(\tau,y)| &= \varphi(1)-\varphi(0) = \int_0^1 \varphi'(\theta)\,d\theta = \\ &= \sum_{ij} d^{ij}[g_{ij}(x) - g_{ij}(y) + \lambda_{ij}(t,x) - \lambda_{ij}(\tau,y)], \end{aligned} \tag{8}$$

where

$$d^{ij} = \int_0^1 g^{ij}_\theta\,d\theta. \tag{9}$$

Since matrices $g(x)+\lambda(t,x)$ satisfy condition (5), then the matrix g_θ satisfies (5) as well, and for elements of the inverse matrix we have

$$\frac{1}{c_2}|\xi|^2 \le g^{ij}_\theta \xi_i\xi_j \le \frac{1}{c_1}|\xi|^2.$$

Integrating the above inequality with respect to θ from 0 to 1 we get

$$\frac{1}{c_2}|\xi|^2 \le d^{ij}\xi_i\xi_j \le \frac{1}{c_1}|\xi|^2. \tag{10}$$

Then

$$\begin{aligned} &\ln|g(x)+\lambda(t,x)| - \ln|g(y)+\lambda(\tau,y)| \le \\ &\le \sum_{ij} |d^{ij}|(|g_{ij}(x)-g_{ij}(y)| + |\lambda_{ij}(t,x)-\lambda_{ij}(\tau,y)|) \le \\ &\le \tfrac{1}{c_1}\sum_{ij}(M_g^\alpha|x-y|^\alpha + M_\Lambda^\alpha \rho(z_1,z_2)^\alpha) \le \tfrac{m^2}{c_1}(M_g^\alpha + M_\Lambda^\alpha)\rho^\alpha(z_1,z_2), \end{aligned} \tag{11}$$

where M_g^α is the maximal Hölder constant for g_{ij}.

To obtain estimate for the second term we denote for a while $g(x) = a$, $g(y) = b$ and consider $\ln|s|$ as a function of m^2 variables $s = (s_{ij}) \in S$.

$$|\ln|a| - \ln|b|\,| \le \sum_{ij} \sup_{t\in[0,1]} \frac{\partial \ln|s|}{\partial s_{ij}}(ta+(1-t)b)|b_{ij}-a_{ij}|.$$

Since for any matrix $s = (s_{ij})$ we have $\frac{\partial \ln|s|}{\partial s_{ij}} = s^{ij}$, then

$$|\ln|a| - \ln|b|\,| \le \sum_{ij} \sup_{t\in[0,1]} (ta+(1-t)b)^{ij}|b_{ij}-a_{ij}|.$$

Put $G = \{g(x), x \in \bar{\Omega}'_k\}$ and let $\overline{\text{co}}\,G$ be its convex hull. Let M_g be the bound for elements of matrices that are inverse to matrices from $\overline{\text{co}}\,G$. Then

$$|\ln|g(x)| - \ln|g(y)|| \le M_g M_g^\alpha m^2 |x-y|^\alpha. \tag{12}$$

Combining (11) - (12) we obtain the estimate that we need. □

We shall use equality (8) ones more to obtain the following:

Lemma 2. *Under the assumptions of lemma 1 we have*

$$\sum_{ij} d^{ij}[\lambda_{ij}(t,x) - \lambda_{ij}(\tau,y)] = F(x,\lambda(t,x)) - F(y,\lambda(\tau,y)) + F_1(x,y),$$

where d^{ij} are given by (9) and $F_1(x,y)$ satisfies Hölder condition

$$|F_1(x,y)| \le \hat{M}_g|x-y|^\alpha$$

with $\hat{M}_g = M_g^\alpha(1/c_1 + M_g)m^2$.

Proof. From (8) and(7) we have:

$$\begin{aligned}&\sum_{ij} d^{ij}[\lambda_{ij}(t,x)-\lambda_{ij}(\tau,y)]=\\&\ln|g(x)+\lambda(t,x)|-\ln|g(y)+\lambda(\tau,y)|+\sum_{ij} d^{ij}[g_{ij}(y)-g_{ij}(x)]=\\&F(x,\lambda(t,x))-F(y,\lambda(\tau,y))+[\ln|g(x)|-\ln|g(y)|]+\\&\sum_{ij} d^{ij}[g_{ij}(y)-g_{ij}(x)].\end{aligned}\tag{13}$$

Consider $F_1(x,y)=[\ln|g(x)|-\ln|g(y)|]+\sum_{ij} d^{ij}[g_{ij}(y)-g_{ij}(x)]$. Inequalities (10) and (12) give

$$|F_1(x,y)|\leq M_g^{\alpha}(1/c_1+M_g)m^2|x-y|^{\alpha}\tag{14}$$

□

3. Hölder Estimate for u_t

Theorem 5. *Let $u(t,x)$ be a solution of (1-2) from the space $C\left([0,T],C^4(V)\right)\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Assume that the right hand side $f(t,x,u)$ bounded and has bounded derivatives up to the second order, $f_u(t,x,u)\geq\delta>0$ on $[0,T]\times V\times R^1$. Let u_0 be an admissible function from $C^{2+\alpha}(V)$. Then*

$$|u_t(z_1)-u_t(z_2)|\leq N\rho^{\beta}(z_1,z_2)\tag{15}$$

with some power $\beta\in(0,\alpha]$ depending on dimension m and constants c_1, c_2 from theorem 3. The constant N depends on β, m, c_1 c_2, D, g, $\|u_0\|_{C^{2+\alpha}}$, $\|f\|_{C^2}$, and on δ.

Proof. Fix a number ρ_0, $0<\rho_0<1/2$, we begin with estimate for u_t on the cylinder $[\rho_0,T]\times V$.

Suppose that the manifold V is covered by charts (Ω_k,φ_k) whose images coincide with $B_1(0)$, where $B_r(0)$ is the ball in the Euclidean space R^m of radius r centered at the origin, and preimages Ω'_k of balls $B_{1/2}(0)$ cover V as well. Differentiating (1) in t within local coordinates of chart Ω_k, we have:

$$-\frac{\partial u_t}{\partial t}+g_u^{\alpha\beta}\nabla_{\alpha\beta}u_t=f_t+f_uu_t$$

Write $v = u_t$. We have got a linear equation with respect to v:

$$Lv = f_t, \tag{16}$$

where

$$L = -\partial/\partial t + g_u^{\alpha\beta}\nabla_{\alpha\beta} - f_u$$

is a uniformly parabolic operator due to theorem 3.

By Q and Q_ρ we denote cylinders $Q = (0,T) \times B_1(0)$, $Q_\rho = (\rho, T) \times B_{1/2}(0)$ in R^{m+1}. By $\partial' Q$ denote the parabolic boundary of the cylinder Q: $\partial' Q = \big(\{0\} \times \bar{B}_1(0)\big) \bigcup \big((0,T) \times \partial B_1(0)\big)$. Let $\rho(z, z')$ be the parabolic distance (6) between points $z = (t, x)$, $z' = (t', z')$, for a point $z \in Q$ we write

$$\rho(z) = \inf\{\rho(z, z'), z' = (t', x') \in \partial Q, t' < t\}, \tag{17}$$

$\rho(z)$ is said to be the parabolic distance from z to the boundary of Q.

Note that $\inf\{\rho(z), z \in Q_{\rho_0}\} = \rho_0$.

For a solution $v = u_t$ of uniformly parabolic equation (16) there is the following Hölder estimate ([7],theorem IV.2.7, p.120): for $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2)$, $z_1, z_2 \in Q_{\rho_0}$,

$$|u_t(z_1) - u_t(z_2)| \leq N(\sup_Q |u_t| + \|Lu_t\|_{L_{m+1}(Q)})\rho^\gamma(z_1, z_2)$$

with some power $\gamma \in (0,1)$, depending on m and constants c_1, c_2 from theorem 3. The constant N depends on m, c_1 c_2 as well, and extra on $\sup|f_u|$ and distance ρ_0 from the parabolic boundary.

Using the estimate of $|u_t|$ (theorem 2) and the equality $Lu_t = f_t$, we get

$$|u_t(z_1) - u_t(z_2)| \leq N_1 \rho^\gamma(z_1, z_2) \tag{18}$$

with N_1 depending on $m, c_1, c_2, \delta, \rho_0, D$, metric tensor g, initial function u_0, right hand side f, and their derivatives up to the second order.

Before getting an estimate of $v = u_t$ for small $t \in (0, \rho_0)$, let us consider the case $t = 0$.

If $t = 0$, then from equation (1) and initial condition (2) we have

$$u_t(0,x) = \ln M(u)(0,x) - f(0,x,u_0). \tag{19}$$

The initial function $u_0 \in C^{2+\alpha}(V)$ defines the continuous section $\Lambda_0 \colon \bar{\Omega}'_k \to \mathbf{S}$ of the bundle $\pi \colon \mathbf{S} \to \bar{\Omega}'_k$ as follows: $\Lambda_0(x) = (x, \nabla_{ij} u_0(x))$. For the section Λ_0 we have constants c_1 and c_2 in (5) are equal to the minimal and maximal eigenvalues of matrices $(g_{ij}(x) + \nabla_{ij} u_0(x))$ and depend on the initial metric g and second order derivatives of the initial function.

Application of lemma 1 gives

$$|\ln M(u_0)(x) - \ln M(u_0)(y)| \le (N_1 + N_2)|x-y|^\alpha,$$

where $N = N_1 + N_2$ depends on m, $\|g\|_{C^{2+\alpha}}, \|u_0\|_{C^{2+\alpha}}$.

On the other hand,

$$\begin{aligned}
&|f(0,x,u_0(x)) - f(0,y,u_0(y))| = \\
&\left|\sum_{i=1}^m \frac{\partial f}{\partial x^i}(0, \theta x + (1-\theta)y, \theta u_0(x) + (1-\theta)u_0(y))(x^i - y^i) + \right. \\
&f_u(0, \theta x_1 + (1-\theta)x_2, \theta u_0(x_1) + (1-\theta)u_0(x_2))(u_0(x_1) - u_0(x_2))| \le \\
&\sup \left|\frac{\partial f}{\partial x^i}\right| |x-y| + \sup |f_u| \sup |\tfrac{\partial u_0}{\partial x^i}||x-y|.
\end{aligned}$$

Thus from (19) we have

$$|u_t(0,x) - u_t(0,y)| \le N_0 |x-y|^\alpha,$$

where α is the Hölder power of u_0 and N_0 depends on m, $\|g\|_{C^{2+\alpha}}, \|u_0\|_{C^{2+\alpha}}$, and first order derivatives of f.

To estimate u_t on the cylinder $(0, \rho_0) \times V$ we use another theorem ([7],th. IV.4.5, p.142). Choose a covering (Ω_k, φ_k) of V such that images of Ω_k in the space R^m coincide with balls of radius $r = 3\sqrt{2}$ centered at $(3, 0, \ldots, 0) \in R^m$ and preimages Ω'_k of sets $\{(x_1, \ldots, x_m) : 1/2 < x_1 < 2, |x_i| < 1, i = 2, \cdots, m\}$ cover V as well. Then we apply the theorem mentioned above to uniformly parabolic equation (16). It claims existence of a constant $\tilde{\gamma}_0 \in (0,1)$, $\tilde{\gamma}_0 \le \alpha$, depending on m, c_1 c_2 such that for every $\tilde{\gamma} \in (0, \tilde{\gamma}_0]$ we have the following estimate

$$|u(z_1) - u(z_2)| \le \rho^{\tilde{\gamma}}(z_1, z_2)(M_{\tilde{\gamma}} + M_2 + M_1 q^{-\tilde{\gamma}})N \tag{20}$$

with constant N depending on α, m, c_1, c_2, $\sup|f_u|$, where $M_{\tilde{\gamma}}$ is the Hölder constant for $u_t(0,x)$ on the lower base $t=0$, which corresponds to the power $\tilde{\gamma}$, M_2 is the bound for the right hand side f_t of equation (16), M_1 is a constant from theorem 2, and $q=1/2$ due to the choice of charts.

Then inequalities (18) and (20) give the estimate that we need on the hole cylinder $[0,T]\times V$ with power $\beta=\min\{\gamma,\tilde{\gamma}\}$. □

4. **Hölder Estimates for $\nabla_{ij}u$**

Theorem 6. *Let $u(t,x)$ be a solution of (1-2) from the space $C\left([0,T],C^4(V)\right)$ $\bigcap C^1\left([0,T],C^2(V)\right)\bigcap C^2\left([0,T],C(V)\right)$. Assume that the right hand side $f(t,x,u)$ is bounded and has bounded derivatives up to the second order, $f_u(t,x,u)\geq\delta>0$ on $[0,T]\times V\times R^1$. Suppose that u_0 is an admissible function and belongs to $C^{2+\alpha}(V)$. Then*

$$|\nabla_{ij}u(z_1)-\nabla_{ij}u(z_2)|\leq N\rho^{\beta}(z_1,z_2) \tag{21}$$

with some power $\beta\in(0,\alpha]$ depending on m and constants c_1, c_2 from theorem 3. The constant N depends on β, m, c_1 c_2, diameter D, metric g, $\|u_0\|_{C^{2+\alpha}}$, $\|f\|_{C^2}$, and δ.

Proof. Suppose that V is covered by local charts in the same way as in the proof of theorem 5. Let $z=(t,x)$ be a fixed point in $(0,T]\times\varphi_k(\Omega_k)$. Let γ be an arbitrary direction in the model space. Differentiating (1) with respect to γ, we have:

$$-\frac{\partial}{\partial t}\nabla_\gamma u+g_u^{\alpha\beta}\nabla_{\gamma\alpha\beta}u=f_\gamma+f_u\nabla_\gamma u.$$

Differentiating once more, we get

$$\begin{aligned}-\tfrac{\partial}{\partial t}\nabla_{\gamma\gamma}u\quad &+\nabla_\gamma(g_u^{\alpha\beta})\nabla_{\gamma\alpha\beta}u+g_u^{\alpha\beta}\nabla_{\gamma\gamma\alpha\beta}u=\\ &=\nabla_\gamma(f_\gamma)+\nabla_\gamma(f_u)\nabla_\gamma u+f_u\nabla_{\gamma\gamma}u.\end{aligned}$$

Write $w=\nabla_{\gamma\gamma}u$, then

$$\begin{aligned}-w_t\;-\;&g_u^{\alpha k}g_u^{l\beta}\nabla_{\gamma kl}u\nabla_{\gamma\alpha\beta}u+g_u^{\alpha\beta}\nabla_{\alpha\beta}w+E\\ &=f_{\gamma\gamma}+2f_{u\gamma}\nabla_\gamma u+f_{uu}(\nabla_\gamma u)^2+f_u w,\end{aligned} \tag{22}$$

where $E = g_u^{\alpha\beta}(\nabla_{\gamma\gamma\alpha\beta}u - \nabla_{\alpha\beta\gamma\gamma}u)$. Commutation formulas for forth order covariant derivatives, which contain coefficients of the curvature tensor and second order covariant derivatives, imply the following estimate ([5], lemma 2):

$$|E| \leq [a(m - \Delta u) + b]g_u^{\lambda\mu}g_{\lambda\mu} + c, \tag{23}$$

where a, b, c are positive constants, depending on diameter and curvature tensor of V. Using the estimate of $(m - \Delta u)$ (theorem 2) and uniformly equivalence of metrics g_u (theorem 3), we obtain

$$|E| \leq \frac{1}{c_1}(aK + b)g^{\lambda\mu}g_{\lambda\mu} + c = \frac{1}{c_1}(aK + b)m + c \overset{def}{=} M.$$

Let $u(t, x)$ be a solution of (1-2). Denote by L the linear differential operator $Lw = -w_t + g_u^{\alpha\beta}\nabla_{\alpha\beta}w - f_u w$. Coefficients $g_u^{\alpha\beta}$ at higher order derivatives continuous if the solution $u(t, x)$ has continuous derivatives with respect to spatial variables up to the second order. The second term in (22) nonnegative since F is convex. Therefore we get the following linear differential inequality :

$$Lw \geq -E + f_{\gamma\gamma} + 2f_{u\gamma}\nabla_\gamma u + f_{uu}(\nabla_\gamma u)^2.$$

Second order derivatives of the right hand side f are bounded, and we have the estimate $|\nabla_\gamma u| \leq 2D$, thus we obtain the inequality

$$Lw \geq -K_1, \tag{24}$$

with a constant $K_1 > 0$ depends on diameter and curvature tensor of V, and on $\|f\|_{C^2}$.

We are going to use Hölder estimates for solutions of a system of uniformly parabolic inequalities ([7]), but we need one more inequality. It will be obtained separately for interior points and for points near the base $\{0\} \times V$ of cylinder. Fix a number ρ_0, $0 < \rho_0 < 1/2$, and choose cylinders Q and Q_ρ as in theorem 5.

Each solution $u(t, x)$ of (1) is an admissible function and determine the continuous solution Λ_u of the bundle $\bar{\mathbf{S}}$:

$$\Lambda_u(t, x) = (t, x, \lambda_u(t, x)) = (t, x, \nabla_{ij}u(t, x)).$$

The above section satisfies condition (5) (theorem 3) with c_1, c_2 depending on diameter of V, metric g, curvature tensor, $\|f\|_{C^2}$, δ, and $\|u_0\|_{C^2}$. Lemma 2 implies

$$\sum_{ij} d^{ij}[\nabla_{ij}u(t,x) - \nabla_{ij}u(\tau,y)] = F(x,\lambda_u(t,x)) - F(y,\lambda_u(\tau,y)) + F_1(x,y), \quad (25)$$

where $F_1(x,y)$ satisfies Hölder's condition with power α and Hölder's constant $\hat{M}_g = M_g^\alpha(1/c_1 + M_g)m^2$. In (25) we have

$$F(x,\lambda_u(t,x)) = F(x,\nabla_{ij}u(t,x)) = \ln M(u)(t,x), \quad (26)$$

Let us write equation (1) at points $z = (t,x), z' = (\tau,y) \in Q_{\rho_0}$:

$$-u_t(t,x) + \ln M(u)(t,x) = f(t,x,u(t,x)),$$

$$-u_t(\tau,y) + \ln M(u)(\tau,y) = f(\tau,y,u(\tau,y)).$$

Subtracting yields:

$$\begin{aligned}&\ln M(u)(t,x) - \ln M(u)(\tau,y) =\\ &[u_t(t,x) - u_t(\tau,y)] + [f(t,x,u(t,x)) - f(\tau,y,u(\tau,y)].\end{aligned}$$

Then using the Hölder estimate for u_t (theorem 5), mean value theorem for $f(t,x,u)$ regarded as a function of three variables, and estimates from theorems 1, 2, we get

$$\begin{aligned}&|\ln M(u)(t,x) - \ln M(u)(\tau,y)| \le |u_t(t,x) - u_t(\tau,y)| +\\ &|f(t,x,u(t,x)) - f(\tau,y,u(\tau,y)| \le N\rho^\beta(z_1,z_2) +\\ &\sup|f_t||t-\tau| + \sup|\nabla_x f||x-y| + \sup|f_u||u(t,x) - u(\tau,y)| \le\\ &N_1\rho^\beta(z_1,z_2),\end{aligned} \quad (27)$$

with β is Hölder's power for u_t; N_1 depends on β, m, c_1 c_2, D, g, $\|u_0\|_{C^{2+\alpha}}$, $\|f\|_{C^2}$, and δ.

Therefore from (25), (26), and (27) we have

$$\begin{aligned}&\sum_{ij} d^{ij}(\nabla_{ij}u(t,x) - \nabla_{ij}u(\tau,y)) \le\\ &N_1\rho^\beta(z_1,z_2) + \hat{M}_g|x-y|^\alpha \le N_2\rho^\beta(z_1,z_2).\end{aligned} \quad (28)$$

where N_2 depends on the same values as N_1 and Hölder's constant of coefficients of g.

Now we use lemma from [7](p.212, lemma V.5.4)(see also [8], lemma 5.2, for another wording). It claims that for all positive definite matrices (a_{ij}) satisfying the condition

$$d_1|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq d_2|\xi|^2, \tag{29}$$

there exists a natural number n, unit vectors $\gamma_1, \cdots, \gamma_n$, and $d \in (0,1)$, depending on m, d_1, d_2, such that the following inequality holds

$$a_{ij}u_{ij} \geq d\sum_{i=1}^{n}(u_{\gamma_i\gamma_i})_+ - 3d_2\sum_{i=1}^{n}(u_{\gamma_i\gamma_i})_-, \tag{30}$$

where u_{γ_i} denotes the derivative in the direction of vector γ_i and $c_+ = \max\{0, c\}$, $c_- = \max\{0, -c\}$.

The above claim contains partial derivatives, but it is true for covariant derivatives due to linearity of the covariant derivative with respect to subscript vector field. Indeed, inequality (30) is based on presentation of a matrix $A = (a_{ij})$ in the form $A = \sum\limits_{i=1}^{n}\beta_i(A)\gamma_i \otimes \gamma_i$, which implies presentation of a linear operator: $Lu = \mathrm{tr}(A \cdot D^2u) = \sum\limits_{i=1}^{n}\beta_i(A)\gamma_i^k\gamma_i^l\nabla_{kl}u$. Here $\gamma_i^k\gamma_i^l\nabla_{kl}u = \nabla_{\gamma_i\gamma_j}u$. Indeed, let γ be a direction, $\gamma = \gamma_k\partial_k$, where γ_k are constant coefficients. Then $\nabla_\gamma u = \nabla_{\gamma_1\partial_k+\cdots+\gamma_m\partial_m}u = \sum\limits_k \gamma_k\nabla_k u$ and $\nabla_{\gamma\gamma}u = \nabla_\gamma(\sum\limits_k \gamma_k u_k) = \sum\limits_k \gamma_k\nabla_\gamma(\nabla_k u) = \sum\limits_{k,l}\gamma_k\gamma_l\nabla_l(\nabla_k u) = \sum\limits_{k,l}\gamma_k\gamma_l\nabla_{lk}u = \sum\limits_{k,l}\gamma_k\gamma_l\nabla_{kl}u$.

Applying (28) to $d^{ij}\nabla_{ij}u(t,x)$ and $d^{ij}\nabla_{ij}(-u(\tau,y))$, and note that $(-c)_+ = c_-$, $(-c)_- = c_+$ we get:

$$\begin{aligned}&\sum_{ij}d^{ij}(\nabla_{ij}u(t,x) - \nabla_{ij}u(\tau,y)) \geq \\ &d\sum_{i=1}^{n}((\nabla_{\gamma_i\gamma_i}u(t,x) - \nabla_{\gamma_i\gamma_i}u(\tau,y)))_+ - \tfrac{3}{c_1}\sum_{i=1}^{n}((\nabla_{\gamma_i\gamma_i}u(t,x) - \nabla_{\gamma_i\gamma_i}u(\tau,y)))_-.\end{aligned}$$

Write $w_i = \nabla_{\gamma_i\gamma_i}u$. The above inequality together with (28) imply:

$$\begin{aligned}&\tfrac{N_2c_1}{3}\rho^\beta(z_1,z_2) \geq \\ &\tfrac{dc_1}{3}\sum_{i=1}^{n}(w_i(t,x) - w_i(\tau,y))_+ - \sum_{i=1}^{n}(w_i(t,x) - w_i(\tau,y))_-.\end{aligned} \tag{31}$$

Therefore, for every point $z = (t,x)$ in the fixed local chart we have uniformly parabolic inequality (24) and for all $z = (t,x), z' = (\tau,y) \in Q_{\rho_0}$ inequality (31). Put

$K_2 = \max\{K_1, N_2c_1/3\}$, then we shall consider the same constant K_2 in the right hand sides of both inequalities (24), (31).

Now we are ready to use theorem [7](. IV.3.1, .122), which gives Hölder estimates for solutions of system of linear parabolic inequalities. In this theorem we take $f_i(r) \equiv r$ and $\nu = \alpha = \beta$, where β is the Hólder power for u_t (theorem 5). The theorem mentioned above claims existence of power $\beta_0 \in (0,1)$, depending on n, d, m, c_1, c_2, such that for all $\beta' \leq \min\{\beta_0, \beta\}$ and all $z_1, z_2 \in Q_{\rho_0}$ the following inequality holds

$$\sum_{i=1}^{m} |w_i(z_1) - w_i(z_2)| \leq \tilde{\rho}^{-\beta'} \rho^{\beta'}(z_1, z_2) N(K_2\tilde{\rho}^{\beta} + \sum_{i=1}^{m} \sup_Q |w_i|), \tag{32}$$

where $\tilde{\rho} = \min\{\rho(z_1), \rho(z_2), 1\}$ and $\rho(z)$ is the parabolic distance from z to the boundary Q_{ρ_0} (evidently $\tilde{\rho} \geq \rho_0$). The constant N depends on the same values as β_0 and extra on $\sup |f_u|$, β.

Thus substituting ρ_0 for $\tilde{\rho}$ in denominator and 1 for $\tilde{\rho}$ in numerator we get the estimate on $[\rho_0, T] \times V$:

$$\sum_{i=1}^{m} |w_i(z_1) - w_i(z_2)| \leq {\rho_0}^{-\beta'} \rho^{\beta'}(z_1, z_2) N_1, \tag{33}$$

with N_1 depending on diameter, $\|g\|_{C^{0+\alpha}}$, curvature tensor, $\|f\|_{C^2}$, δ, $\|u_0\|_{C^{2+\alpha}}$ and β, where β is the Hölder power for u_t.

To obtain the estimate on $[0, \rho_0) \times V$ we use ([7], theorem IV.5.1, p.147). Proceeding in the same way as in theorem 5 we cover V with charts (Ω_k, φ_k) such that images of Ω_k in the space R^m coincide with balls of radius $r = 3\sqrt{2}$ centered at $(3, 0, \ldots, 0) \in R^m$ and preimages Ω'_k of sets $\{(x_1, \ldots, x_m) : 1/2 < x_1 < 2, |x_i| < 1, i = 2, \cdots, m\}$ cover V as well. Then the theorem mentioned above claims that inequalities (24) and (31) imply existence of a constant $\tilde{\gamma}_0 \in (0, 1)$, depending on n, m, c_1, c_2, d, such that for every $\tilde{\gamma} \in (0, \min\{\tilde{\gamma}_0, \beta\}]$ the following inequality holds

$$\sum_{i=1}^{n} |w_i(z_1) - w_i(z_2)| \leq \rho^{\tilde{\gamma}}(z_1, z_2)(M_{\tilde{\gamma}} + K_2 + Mq^{-\tilde{\gamma}})N \tag{34}$$

with N depending on n, m, c_1, c_2, $\sup|f_u|$, γ, β, where $M_{\tilde{\gamma}}$ is the largest Hólder constant of $\nabla_{\gamma_i\gamma_i}u_0$ with power $\tilde{\gamma}$, $M = \sup \nabla_{\gamma_i\gamma_i}u$, and $q = 1/2$ due to the choice of charts. □

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