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A NEW DIFFERENTIAL INEQUALITY II

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Abstract. We find conditions on the complex-valued functions A and B, in the unit disc U such that the differential inequality

$$Re\,[A(z)p^2(z)+B(z)p(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}-$$
$$-\gamma(zp'(z))^{2k-2}+\delta(zp'(z))^{2k-3}+\eta]>0$$

implies $Re\,p(z)>0$, where $\alpha,\beta,\gamma,\delta\geq 0$, $\eta<\frac{\alpha n^{2k}}{2^{2k}}+\frac{\beta n^{2k-1}}{2^{2k-1}}+\frac{\gamma n^{2k-2}}{2^{2k-2}}+\frac{\delta n^{2k-3}}{2^{2k-3}}$, $p\in\mathcal{H}[1,n]$ and $k\in\mathbb{N},k\geq 2$

1. Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U=\{z\in\mathbb{C}:\quad |\,z\,|<1\}.$$

For $a\in\mathbb{C}$ and $n\in\mathbb{N}^*$ we let

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}[U],\quad f(z)=a+a_nz^n+a_{n+1}z^{n+1}+...,\ z\in U\}$$

and

$$\mathcal{A}_n=\{f\in\mathcal{H}[U],\quad f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+...,\ z\in U\}$$

with $\mathcal{A}_1=\mathcal{A}$.

In order to prove the new results we shall use the following lemma, which is a particular form of Theorem 2.3.i[1,p.35].

Lemma A. [1,p.35] *Let $\psi:\mathbb{C}^2\times U\to\mathbb{C}$ a function which satisfies*

$$Re\ \psi(\rho i,\sigma;z)\leq 0,$$

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where $\rho, \sigma \in \mathbb{R}, \quad \sigma \leq -\frac{n}{2}(1+\rho^2) \quad z \in U \quad$ *and* $\quad n \geq 1$.

If $p \in \mathcal{H}[1,n]$ and

$$Re\, \psi\left(p(z), zp'(z); z\right) > 0$$

then

$$Re\, p(z) > 0.$$

2. **Main results**

Theorem 1. *Let* $\alpha, \beta, \gamma, \delta \geq 0$, $\eta < \frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\alpha n^{2k}}{2^{2k}} 2k - \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) -$$

$$-\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii)(Im B(z))^2 \leq 4\left(\frac{\alpha n^{2k}}{2^{2k}} 2k + \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + \right. \tag{1}$$

$$\left. + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re\, A(z)\right) \cdot$$

$$\cdot \left(\frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1,n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \alpha(zp'(z))^{2k} + \beta(zp'(z))^{2k-1} -$$

$$-\gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \tag{2}$$

then

$$Re\, p(z) > 0.$$

Proof. We let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be defined by

$$\psi(p(z), zp'(z); z) = A(z)p^2(z) + B(z)p(z) - \alpha(zp'(z))^{2k} +$$

$$+\beta(zp'(z))^{2k-1} - \gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta \tag{3}$$

From (2) we have

$$Re\,\psi(p(z), zp'(z); z) > 0 \text{ for } z \in U. \tag{4}$$

For $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \le -\frac{n}{2}(1+\rho^2)$, hence

$$-\sigma^{2k} \le -\frac{n^{2k}}{2^{2k}}(1+\rho^2)^{2k};\ \sigma^{2k-1} \le -\frac{n^{2k-1}}{2^{2k-1}}(1+\rho^2)^{2k-1}$$

$$-\sigma^{2k-2} \le -\frac{n^{2k-2}}{2^{2k-2}}(1+\rho^2)^{2k-2};\ \sigma^{2k-3} \le -\frac{n^{2k-3}}{2^{2k-3}}(1+\rho^2)^{2k-3}$$

and $z \in U$, by using (1) we obtain

$$Re\,\psi(\rho i, \sigma; z) = Re\,[A(z)(\rho i)^2 + B(z)\rho i - \alpha\sigma^{2k} +$$

$$+\beta\sigma^{2k-1} - \gamma\sigma^{2k-2} + \delta\sigma^{2k-3} + \eta] =$$

$$= -\rho^2 Re\,A(z) - \rho\, Im\,B(z) - \alpha\sigma^{2k} + \beta\sigma^{2k-1} - \gamma\sigma^{2k-2} + \delta\sigma^{2k-3} + \eta \le$$

$$\le -\rho^2 Re\,A(z) - \rho\, Im\,B(z) - \frac{\alpha n^{2k}}{2^{2k}}(1+\rho^2)^{2k} - \frac{\beta n^{2k-1}}{2^{2k-1}}(1+\rho^2)^{2k-1} -$$

$$-\frac{\gamma n^{2k-2}}{2^{2k-2}}(1+\rho^2)^{2k-2} - \frac{\delta n^{2k-3}}{2^{2k-3}}(1+\rho^2)^{2k-3} + \eta =$$

$$= -\frac{\alpha n^{2k}}{2^{2k}}(\rho^2)^{2k} - \left(\frac{\alpha n^{2k}}{2^{2k}}\mathrm{C}_{2k}^{2k-1} + \frac{\beta n^{2k-1}}{2^{2k-1}}\mathrm{C}_{2k-1}^{2k-1}\right)(\rho^2)^{2k-1} -$$

$$-\left(\frac{\alpha n^{2k}}{2^{2k}}\mathrm{C}_{2k}^{2k-2} + \frac{\beta n^{2k-1}}{2^{2k-1}}\mathrm{C}_{2k-1}^{2k-2} + \frac{\gamma n^{2k-2}}{2^{2k-2}}\mathrm{C}_{2k-2}^{2k-2}\right)(\rho^2)^{2k-2} - \ldots -$$

$$-\left[\left(\frac{\alpha n^{2k}}{2^{2k}}2k + \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \right.\right.$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re\,A(z)\right)\rho^2 + \rho\, Im\,B(z) +$$

$$\left. + \frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right] \le 0.$$

By using Lemma A we have that $Re\,p(z) > 0$. □

If $\alpha = 0$ and $\beta = 0$, then Theorem 1 can be rewritten as follows:

Corollary 1. *Let* $\gamma, \delta \geq 0$, $\eta < \dfrac{\gamma n^{2k-2}}{2^{2k-2}} + \dfrac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii)(ImB(z))^2 \leq 4\left(\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + ReA(z)\right) \tag{5}$$

$$\cdot \left(\frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \tag{6}$$

then

$$Re\, p(z) > 0.$$

Remarks. 1. This result from Corollary 1 was obtained in Theorem 1 from [2].

2. For $\alpha = 0$, $\quad k = 2$ we obtain Theorem 1 from [3].

If $\alpha = 0$, then Theorem 1 can be rewritten as follows:

Corollary 2. *Let* $\beta, \gamma, \delta \geq 0$, $\eta < \dfrac{\beta n^{2k-1}}{2^{2k-1}} + \dfrac{\gamma n^{2k-2}}{2^{2k-2}} + \dfrac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* n, k *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) - \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - $$

$$-\frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii)(ImB(z))^2 \leq 4\left(\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2)\right. \tag{7}$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + ReA(z)\right)\left(\frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) + \beta(zp'(z))^{2k-1} -$$

$$-\gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \tag{8}$$

then

$$Re\, p(z) > 0.$$

If $\beta = 0$, then Theorem 1 can be rewritten as follows.

Corollary 3. *Let* $\alpha, \gamma, \delta \geq 0$, $\eta < \frac{\alpha n^{2k}}{2^{2k}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* $n,\ k$ *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\alpha n^{2k}}{2^{2k}} 2k - \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii) (Im B(z))^2 \leq 4\left(\frac{\alpha n^{2k}}{2^{2k}} 2k + \frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2) + \right. \tag{9}$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re\, A(z)\right)\left(\frac{\alpha n^{2k}}{2^{2k}} + \frac{\gamma n^{2k-2}}{2^{2k-2}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \alpha(zp'(z))^{2k} -$$

$$-\gamma(zp'(z))^{2k-2} + \delta(zp'(z))^{2k-3} + \eta] > 0 \tag{10}$$

then

$$Re\, p(z) > 0.$$

If $\gamma = 0$, then Theorem 1 can be rewritten as follows.

Corollary 4. *Let* $\alpha, \beta, \delta \geq 0$, $\eta < \frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\delta n^{2k-3}}{2^{2k-3}}$ *and let* $n,\ k$ *be positive integers,* $k \geq 2$. *Suppose that the functions* $A, B : U \to \mathbb{C}$ *satisfy*

$$(i) Re\, A(z) > -\frac{\alpha n^{2k}}{2^{2k}} 2k - \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) - \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3)$$

$$(ii) (Im B(z))^2 \leq 4\left(\frac{\alpha n^{2k}}{2^{2k}} 2k + \frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1) + \right. \tag{11}$$

$$\left. + \frac{\delta n^{2k-3}}{2^{2k-3}}(2k-3) + Re A(z)\right)\left(\frac{\alpha n^{2k}}{2^{2k}} + \frac{\beta n^{2k-1}}{2^{2k-1}} + \frac{\delta n^{2k-3}}{2^{2k-3}} - \eta\right)$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Re\, [A(z)p^2(z) + B(z)p(z) - \alpha(zp'(z))^{2k} + \beta(zp'(z))^{2k-1} -$$

$$+\delta(zp'(z))^{2k-3}+\eta]>0 \tag{12}$$

then

$$Re\, p(z)>0.$$

If $\delta=0$, then Theorem 1 can be rewritten as follows:

Corollary 5. *Let* $\alpha,\beta,\gamma\geq 0$, $\eta<\frac{\alpha n^{2k}}{2^{2k}}+\frac{\beta n^{2k-1}}{2^{2k-1}}+\frac{\gamma n^{2k-2}}{2^{2k-2}}$ *and let* n, k *be positive integers,* $k\geq 2$*. Suppose that the functions* $A,B:U\to\mathbb{C}$ *satisfy*

$$(i) Re\, A(z)>-\frac{\alpha n^{2k}}{2^{2k}}2k-\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1)-\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2)$$

$$(ii)(Im B(z))^2\leq 4\left(\frac{\alpha n^{2k}}{2^{2k}}2k+\frac{\beta n^{2k-1}}{2^{2k-1}}(2k-1)+\right. \tag{13}$$

$$\left.+\frac{\gamma n^{2k-2}}{2^{2k-2}}(2k-2)+Re A(z)\right)\left(\frac{\alpha n^{2k}}{2^{2k}}+\frac{\beta n^{2k-1}}{2^{2k-1}}+\frac{\gamma n^{2k-2}}{2^{2k-2}}-\eta\right)$$

If $p\in\mathcal{H}[1,n]$ *and*

$$Re\,[A(z)p^2(z)+B(z)p(z)-\alpha(zp'(z))^{2k}+\beta(zp'(z))^{2k-1}-$$

$$-\gamma(zp'(z))^{2k-2}+\eta]>0 \tag{14}$$

then

$$Re\, p(z)>0.$$

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