STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 4, December 2004

ON THE INVARIANCE PROPERTY OF THE FISHER INFORMATION (I)

CRISTINA-IOANA FĂTU

Abstract. The objective of this paper is to give some properties for the Fisher's information measure when $X_{a\leftrightarrow b}$ represents a bilateral truncated random variable that corresponds to a normal random variable X with the probability density function $f(x;\theta)$, where $\theta = (m, \sigma^2)$, $\theta \in D_\theta$, $D_\theta \subseteq \mathbb{R}^2$, $m \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$.

The Fisher's invariance property will be studied in the case of a truncated normal distribution.

Let X be a normal distribution with probability density function

$$f(x;m,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right\}, x \in \mathbb{R}, \tag{1}$$

where the parameters m and σ have their usual significance, namely: $m = E(X)$, $\sigma^2 = Var(X)$, $m \in \mathbb{R}$, $\sigma > 0$.

Definition 1. [1] *We say that the random variable X has a normal distribution truncated to the left at $X = a, a \in \mathbb{R}$ and to the right at $X = b, b \in \mathbb{R}$, denoted by $X_{a\leftrightarrow b}$, if its probability density function, denoted by $f_{a\leftrightarrow b}(x;m,\sigma^2)$, has the form*

$$f_{a\leftrightarrow b}(x;m,\sigma^2) = \begin{cases} \dfrac{k(a,b)}{\sqrt{2\pi}\sigma} \exp\left\{-\dfrac{1}{2}\left(\dfrac{x-m}{\sigma}\right)^2\right\} & if\ a \le x \le b, \\ 0 & if\ x < a\ or\ x > b, \end{cases} \tag{2}$$

where

$$k(a,b) = \frac{1}{A} = \frac{1}{\Phi\left(\dfrac{b-m}{\sigma}\right) - \Phi\left(\dfrac{a-m}{\sigma}\right)}, \tag{3}$$

Received by the editors: 15.12.2004.

2000 *Mathematics Subject Classification.* 62B10, 62B05.

Key words and phrases. Fisher's information, truncated distribution, invariance property.

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{t^2}{2}\right) dt, \tag{4}$$

$$\Phi(-\infty) = 0,\ \Phi(0) = \frac{1}{2},\ \Phi(+\infty) = 1,\ \Phi(-z) = 1 - \Phi(z), \tag{5}$$

$\Phi(z)$ is the standard normal distribution function corresponding to the standard normal random variable

$$Z = \frac{X - m}{\sigma},\ E(Z) = 0, Var(Z) = 1. \tag{6}$$

The probability density function of the random variable Z has the form

$$f(z;0,1) = f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right), z \in (-\infty, +\infty). \tag{7}$$

Remark 1. *A truncated probability distribution can be regarded as a conditional probability distribution in the sense that if X has an unrestricted distribution with probability density function $f(x)$ then $f_{a\leftrightarrow b}(x)$, as defined above, is the probability density function which governs the behavior of X subject to the condition that X is known to lie in $[a,b]$.*

Theorem 1. [2] *Let $X_{a\leftrightarrow b}$ be a random variable with a normal distribution truncated to the left at $X = a$ and to the right at $X = b$. Then*

$$E(X_{a\leftrightarrow b}) = m - \frac{\sigma^2}{A}\left[f(b;m,\sigma^2) - f(a;m,\sigma^2)\right], \tag{8}$$

where

$$f(a;m,\sigma^2) = f(x;m,\sigma^2)\mid_{x=a} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{a-m}{\sigma}\right)^2\right), \tag{9}$$

$$f(b;m,\sigma^2) = f(x;m,\sigma^2)\mid_{x=b} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{b-m}{\sigma}\right)^2\right). \tag{10}$$

Theorem 2. [2] *Let $X_{a\leftrightarrow b}$ be a random variable with a normal distribution truncated to the left at $X = a$ and to the right at $X = b$. Then*

$$E(X^2_{a\leftrightarrow b}) = m^2 + \sigma^2 - \frac{\sigma^2}{A}\left((m+b)f(b;m,\sigma^2) - (m+a)f(b;m,\sigma^2)\right). \tag{11}$$

Corollary 1. [2] *If $X_{a\leftrightarrow b}$ is a random variable with a normal distribution truncated to the left at $X = a$ and to the right at $X = b$, then*

$$Var(X_{a\leftrightarrow b}) = \sigma^2 + \frac{\left(\sigma^2\right)^2}{A^2}\left(f(b;m,\sigma^2) - f(a;m,\sigma^2)\right)^2 + \tag{12}$$

$$+\frac{\sigma^2}{A}\left((m-b)f(b;m,\sigma^2) - (m-a)f(a;m,\sigma^2)\right). \tag{13}$$

Corollary 2. [1] *For the random variables $X_{a\leftarrow}$, $X_{\rightarrow b}$ and X we have*

$$\lim_{a\to-\infty} f_{a\leftrightarrow b}(x;m,\sigma^2) = f_{\rightarrow b}(x;m,\sigma^2) = \tag{14}$$

$$= \begin{cases} \dfrac{1}{\Phi\left(\dfrac{b-m}{\sigma}\right)}.\ f(x;m,\sigma^2) & if\ x \le b \\ 0 & if\ x > b, \end{cases} \tag{15}$$

$$\lim_{b\to+\infty} f_{a\leftrightarrow b}(x;m,\sigma^2) = f_{a\leftarrow}(x;m,\sigma^2) = \tag{16}$$

$$= \begin{cases} \dfrac{1}{1-\Phi\left(\dfrac{a-m}{\sigma}\right)}.\ f(x;m,\sigma^2) & if\ x \ge a \\ 0 & if\ x < a, \end{cases} \tag{17}$$

and

$$\lim_{\substack{a\to-\infty,\\ b\to+\infty}} f_{a\leftrightarrow b}(x;m,\sigma^2) = f(x;m,\sigma^2) = \tag{18}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right) \quad if\ x \in \mathbb{R}, \tag{19}$$

where $f_{\rightarrow b}(x;m,\sigma^2)$ is the probability density function when $X_{\rightarrow b}$ has a normal distribution truncated to the right at $X = b$; $f_{a\leftarrow}(x;m,\sigma^2)$ is the probability density function when $X_{a\leftarrow}$ has a normal distribution truncated to the left at $X = a$ and $f(x;m,\sigma^2)$ is the probability density function when X has an ordinary normal distribution.

Corollary 3. [1] *For the random variables* $X_{a\leftarrow}$, $X_{\rightarrow b}$ *and* X *we have*

$$E(X_{a\leftarrow}) = \lim_{b\to+\infty} E(X_{a\leftrightarrow b}) = \tag{20}$$

$$= m + \frac{\sigma^2}{1-\Phi\left(\frac{a-m}{\sigma}\right)} f(a; m, \sigma^2), \tag{21}$$

$$E(X_{\rightarrow b}) = \lim_{a\to-\infty} E(X_{a\leftrightarrow b}) = \tag{22}$$

$$= m - \frac{\sigma^2}{\Phi\left(\frac{b-m}{\sigma}\right)} f(b; m, \sigma^2), \tag{23}$$

and

$$E(X) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} E(X_{a\leftrightarrow b}) = \tag{24}$$

$$= m. \tag{25}$$

Corollary 4. [1] *For the random variables* $X_{a\leftarrow}$, $X_{\rightarrow b}$ *and* X *we have*

$$Var(X_{a\leftarrow}) = \lim_{b\to+\infty} Var(X_{a\leftrightarrow b}) = \tag{26}$$

$$= \sigma^2 + \frac{\left(\sigma^2\right)^2 f^2(a; m, \sigma^2)}{\left(1-\Phi(\frac{a-m}{\sigma})\right)^2} - \frac{\sigma^2(m-a) f(a; m, \sigma^2)}{1-\Phi(\frac{a-m}{\sigma})}, \tag{27}$$

$$Var(X_{\rightarrow b}) = \lim_{a\to-\infty} Var(X_{a\leftrightarrow b}) = \tag{28}$$

$$= \sigma^2 + \frac{\left(\sigma^2\right)^2 f^2(b; m, \sigma^2)}{\Phi^2(\frac{b-m}{\sigma})} + \frac{\sigma^2(m-b) f(b; m, \sigma^2)}{\Phi(\frac{b-m}{\sigma})}, \tag{29}$$

and

$$Var(X) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} Var(X_{a\leftrightarrow b}) = \sigma^2. \tag{30}$$

Let consider the case: $m -$ *an unknown parameter*, $\sigma^2 -$ *a known parameter*.

Theorem 3. [2] *If the random variable $X_{a\leftrightarrow b}$ has a bilateral truncated normal distribution, that is, its probability distribution has the form (2), then the Fisher's information measure, about the unknown parameter m, has the following form*

$$I_{X_{a\leftrightarrow b}}(m) = \int_a^b \left(\frac{\partial \ln f_{a\leftrightarrow b}(x;m,\sigma^2)}{\partial m}\right)^2 f_{a\leftrightarrow b}(x;m,\sigma^2)dx = \tag{31}$$

$$= \frac{1}{\sigma^2} - \frac{[f(b;m,\sigma^2) - f(a;m,\sigma^2)]^2}{\sqrt{2\pi}\sigma A^2} +$$

$$+ \frac{(m-b)f(b;m,\sigma^2) - (m-a)f(a;m,\sigma^2)}{\sigma^2 A}, \tag{32}$$

where $f(a;m,\sigma^2)$ and $f(b;m,\sigma^2)$ are given in (9) and (10).

Corollary 5. *If $a = m-\sigma$, $b = m+\sigma$, then the Fisher's information measure, relative to the unknown parameter m, has the following value*

$$I_{X_{m-\sigma\leftrightarrow m+\sigma}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{1}{0,341\sqrt{2\pi e}}\right), \tag{33}$$

moreover, we obtain the inequality

$$I_{X_{m-\sigma\leftrightarrow m+\sigma}}(m) < I_X(m). \tag{34}$$

Corollary 6. *(Invariance of the Fisher information - the first form) If we consider values $a = m$, $b = m+\sigma$ or $a = m-\sigma$, $b = m$, then the Fisher's information measures, relative to the unknown parameter m, has the same value, namely*

$$I_{X_{m\leftrightarrow m+\sigma}}(m) = I_{X_{m-\sigma\leftrightarrow m}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{(1-\sqrt{e})^2}{\left(\sqrt{2\pi e}.0,341\right)^2} + \frac{1}{\sqrt{2\pi e}.0,341}\right)\right\}, \tag{35}$$

moreover, we have the following inequality

$$I_{X_{m\leftrightarrow m+\sigma}}(m) = I_{X_{m-\sigma\leftrightarrow m}}(m) < I_X(m). \tag{36}$$

Corollary 7. *If $a = m - k\sigma$, $b = m + k\sigma$, $k \in \mathbb{N}^*$, then the Fisher's information measure, relative to the unknown parameter m, can be written like*

$$I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(m) = \frac{1}{\sigma^2}\left\{1 - \frac{2k}{\sqrt{2\pi e^{k^2}}(2\Phi(k)-1)}\right\},\ k \in \mathbb{N}^*, \tag{37}$$

moreover, we obtain the inequality

$$I_{X_{m-k\sigma\leftrightarrow m+k\sigma}}(m) < \frac{1}{\sigma^2} = I_X(m),\ k \in \mathbb{N}^*. \tag{38}$$

Remark 2. *In the particular case $k = 3$ we obtain a bilateral truncated random variable $X_{m-k\sigma\leftrightarrow m+k\sigma}$ and the Fisher's information measure, relative to the unknown parameter m, can be written like*

$$I_{X_{m-3\sigma\leftrightarrow m+3\sigma}}(m) = \frac{1}{\sigma^2}\left[1 - \frac{1}{\sqrt{2\pi e}e^4.0,166}\right], \tag{39}$$

moreover, we obtain the inequality

$$I_{X_{m-3\sigma\leftrightarrow m+3\sigma}}(m) < \frac{1}{\sigma^2} = I_X(m). \tag{40}$$

Corollary 8. *For the random variables $X_{a\leftarrow}$, $X_{\rightarrow b}$ and X the Fisher's information measures have the following forms*

$$I_{X_{a\leftarrow}}(m) = \lim_{b\to+\infty} I_{X_{a\leftrightarrow b}}(m) = \tag{41}$$

$$= \frac{1}{\sigma^2} - \frac{(m-a)f(a;m,\sigma^2)}{\sigma^2\left(1-\Phi\left(\frac{a-m}{\sigma}\right)\right)} - \frac{.f^2(a;m,\sigma^2)}{\left(1-\Phi\left(\frac{a-m}{\sigma}\right)\right)^2}, \tag{42}$$

$$I_{X_{\rightarrow b}}(m) = \lim_{a\to-\infty} I_{X_{a\leftrightarrow b}}(m) = \tag{43}$$

$$=\frac{1}{\sigma^2} + \frac{(m-b)f(b;m,\sigma^2)}{\sigma^2\Phi\left(\frac{b-m}{\sigma}\right)} - \frac{f^2(b;m,\sigma^2)}{\Phi^2\left(\frac{a-m}{\sigma}\right)}, \tag{44}$$

and

$$I_X(m) = \lim_{\substack{a\to-\infty,\\ b\to+\infty}} I_{X_{a\leftrightarrow b}}(m) = \frac{1}{\sigma^2}. \tag{45}$$

Corollary 9. *If $b = m$, then from (5) we obtain $\Phi(0) = \frac{1}{2}$ and from (44) it results the inequality*

$$I_{X_{\to m}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right) < \frac{1}{\sigma^2} = I_X(m). \tag{46}$$

Corollary 10. *If $b = m - \sigma$, then from (5) we obtain*

$$\Phi(-1) = 1 - \Phi(1) = 0,159, \tag{47}$$

and from (44), the following relations

$$I_{X_{\to m-\sigma}}(m) = \frac{1}{\sigma^2}\left(1 + \frac{1}{\sqrt{2\pi e}\Phi(-1)} - \frac{1}{\left(\sqrt{2\pi e}\Phi(-1)\right)^2}\right), \tag{48}$$

moreover, the inequalities

$$I_{X_{\to m}}(m) < I_X(m) < I_{X_{\to m-\sigma}}(m). \tag{49}$$

Corollary 11. *If $b = m + \sigma$, we have the following relations*

$$I_{X_{\to m+\sigma}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\}, \tag{50}$$

moreover, the inequalities

$$I_{X_{\to m+\sigma}}(m) < I_{X_{\to m}}(m) < I_X(m) < I_{X_{\to m-\sigma}}(m). \tag{51}$$

Proof. From (44), it results the equality (50) which imply the inequality

$$I_{X_{\to m+\sigma}}(m) = \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\} < \frac{1}{\sigma^2} = I_X(m). \tag{52}$$

Then, from (49) and (52) it results the inequalities

$$I_{X_{\to m+\sigma}}(m) < I_X(m) < I_{X_{\to m-\sigma}}(m). \tag{53}$$

Now, from (46), the inequality (51) is reduced to the following inequality

$$I_{X_{\to m+\sigma}}(m) < I_{\to m}(m). \tag{54}$$

Using the relations (46) and (50), we observe that this last inequality is equivalent to the inequalities

$$\frac{1}{\sqrt{2\pi e}\Phi(1)}+\frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}<\frac{2}{\sqrt{2\pi e}\Phi(1)}<\frac{2}{\pi},$$

or to the inequality

$$\pi<\sqrt{2\pi e}\Phi(1).$$

This last inequality results using the approximations: $\pi \approx 3,14$, $e \approx 2,72$, $\Phi(1) = 0,841$. □

Corollary 12. *If $a = m$, then from (5) we obtain $\Phi(0) = \frac{1}{2}$ and from (42) it results the inequality*

$$I_{X_{m\leftarrow}}(m)=\frac{1}{\sigma^2}\left(1-\frac{2}{\pi}\right)<\frac{1}{\sigma^2}=I_X(m). \tag{55}$$

Corollary 13. *If $a = m - \sigma$, then from (5) we obtain*

$$1-\Phi(-1)=\Phi(1)=0,841, \tag{56}$$

and from (42) it results the equality

$$I_{X_{m-\sigma\leftarrow}}(m)=\frac{1}{\sigma^2}\left\{1-\left(\frac{1}{\sqrt{2\pi e}\Phi(1)}+\frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\}, \tag{57}$$

moreover, the inequality

$$I_{X_{m-\sigma\leftarrow}}(m)<I_X(m). \tag{58}$$

Corollary 14. *If $a = m + \sigma$, then from (5) we obtain $\Phi(-1) = 0,159$, and from (42) it results the equality*

$$I_{X_{m+\sigma\leftarrow}}(m)=\frac{1}{\sigma^2}\left\{1+\left(\frac{1}{\sqrt{2\pi e}\Phi(-1)}-\frac{1}{\left(\sqrt{2\pi e}\Phi(-1)\right)^2}\right)\right\}, \tag{59}$$

moreover, the inequalities

$$I_{X_{m-\sigma\leftarrow}}(m)<I_{m\leftarrow}(m)<I_X(m)<I_{X_{m+\sigma\leftarrow}}(m) \tag{60}$$

Proof. From the relation (42), we obtain the equality (59) which imply the inequality

$$I_{X_{m+\sigma\leftarrow}}(m) = \frac{1}{\sigma^2}\left\{1 + \left(\frac{1}{\sqrt{2\pi e}\Phi\left(-1\right)} - \frac{1}{\left(\sqrt{2\pi e}\Phi\left(-1\right)\right)^2}\right)\right\} > \frac{1}{\sigma^2} = I_X(m). \quad (61)$$

From (58) and (61) it results the inequalities

$$I_{X_{m-\sigma\leftarrow}}(m) < I_X(m) < I_{X_{m+\sigma\leftarrow}}(m). \quad (62)$$

Now, regarding the inequalities (55) and (62), we observe that the inequality (60) is reduced to the inequality

$$I_{X_{m-\sigma\leftarrow}}(m) < I_{X_{m\leftarrow}}(m). \quad (63)$$

By the relations (55) and (57), we observe that this last inequality is equivalent to the inequality

$$\frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi\left(1\right)} + \frac{1}{\left(\sqrt{2\pi e}\Phi\left(1\right)\right)^2}\right)\right\} < \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right),$$

or to the inequalities

$$\frac{1}{\sqrt{2\pi e}\Phi\left(1\right)} + \frac{1}{\left(\sqrt{2\pi e}\Phi\left(1\right)\right)^2} < \frac{2}{\sqrt{2\pi e}\Phi\left(1\right)} < \frac{2}{\pi}.$$

The last inequality is equivalent to the inequality $\sqrt{2\pi e}\Phi\left(1\right) < \left(\sqrt{2\pi e}\Phi\left(1\right)\right)^2$ which imply the inequality

$$\pi < \sqrt{2\pi e}\Phi\left(1\right). \quad (64)$$

Using the approximations: $\pi \approx 3,14$, $e \approx 2,72$ and $\Phi(1) = 0,841$, the last inequality is true, because

$$\sqrt{2\pi e}\Phi\left(1\right) \approx \sqrt{2 \times 3,14 \times 2,72}.0,841 \approx 4,13.0,841 \approx 3,475.$$

□

The invariance of Fisher's information is ilustrated in the following corollaries.

Corollary 15. *(the second form)*

$$I_{X_{\to m+\sigma}}(m) = I_{X_{-\infty\leftrightarrow m+\sigma}}(m) = \tag{65}$$

$$= \frac{1}{\sigma^2}\left\{1 - \left(\frac{1}{\sqrt{2\pi e}\Phi(1)} + \frac{1}{\left(\sqrt{2\pi e}\Phi(1)\right)^2}\right)\right\} = \tag{66}$$

$$= I_{X_{m-\sigma\leftarrow}}(m) = I_{X_{m-\sigma\leftrightarrow+\infty}}(m). \tag{67}$$

Proof. Using the relations (50) and (57), the proof is obviously. □

Corollary 16. *(the third form)*

$$I_{X_{\to m-\sigma}}(m) = I_{X_{-\infty\leftrightarrow m-\sigma}}(m) = \tag{68}$$

$$= \frac{1}{\sigma^2}\left(1 + \frac{1}{\sqrt{2\pi e}\Phi(-1)} - \frac{1}{\left(\sqrt{2\pi e}\Phi(-1)\right)^2}\right) = I_{X_{m+\sigma\leftrightarrow+\infty}}(m). \tag{69}$$

Proof. Using the relations (48) and (59), the proof is obviously. □

Corollary 17. *(the fourth form)*

$$I_{X_{\to m}}(m) = I_{X_{-\infty\leftrightarrow m}}(m) = \frac{1}{\sigma^2}\left(1 - \frac{2}{\pi}\right) = I_{X_{m\leftarrow}}(m) = I_{X_{m\leftrightarrow+\infty}}. \tag{70}$$

Proof. Using the relations (46) and (55), the proof is obviously. □

References

[1] I. Mihoc, C. I. Fătu, *Fisher's Information Measures for the Truncated Normal Distribution* (I), Analysis, Functional Equations, Approximation and Convexity, Proceedings of the Conference Held in Honour Professor Elena Popoviciu on the Occasion of the 75^{th}Birthday, Editura Carpatica 1999, 171-182.

[2] I. Mihoc, C. I. Fătu, *Fisher's Information Measures and Truncated Normal Distributions* (II), Seminarul de teoria celei mai bune aproximări, convexitate şi optimizare, 1960-2000, ,,Omagiu memoriei academicianului Tiberiu Popoviciu la 25 de ani de la moarte", 26-29 octombrie, 2000, 153-155.

[3] I. Mihoc, C. I. Fătu, *Some Observations about Fisher's information Measures*, "The 130^{-th} Pannonian Applied Mathematical Meeting", July15-22, 2000, Cluj-Napoca-Miercurea-Ciuc (Gábor Dénes-Foundation Cluj) (Interuniversity Network in Central

Europe), Bulletins for Applied & Computer Mathematics, Caretaked by the PAMM-Center at the Technical University of Budapest, Budapest, 2000, BAM-1788, 183-193.

[4] I. Mihoc, C. I. Fătu, *Calculul probabilităţilor şi statistică matematică*, Casa de Editură Transilvania Pres, Cluj-Napoca, 2003.

[5] C. R. Rao, *Liniar Statistical Inference and Its Applications*, John Wiley and Sons, Inc., New York, 1965.

[6] A. Rényi, *Probability Theory*, Akadémiai Kiado, Budapest, 1970.

[7] M. J. Schervish, *Theory of Statistics*, Springer-Verlag New York, Inc. 1995

Faculty of Economical Sciences at Christian University,
"Dimitrie Cantemir", 3400, Cluj-Napoca, Romania
E-mail address: cfatu@cantemir.cluj.astral.ro