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INTEGRAL $\lambda - \tau$ BIVARIATE SPLINE OPERATORS IN COMPUTER GRAPHICS PROBLEMS

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Abstract. In the present work we propose and analyze a particular class of bivariate tensor VDS splines defined by an integral operator and depending on two shape parameters (λ and τ). These functions are used to generate surface models. Precisely we generate and algebrically formalize a $\lambda - \tau$ parametric integral spline family and advocate its use in the field of computer graphics. We apply such models to the problem of reconstructing, starting from a set of measured points, "smooth" surfaces (where the optimal value of the shape parameters is obtained minimizing suitable functionals).

Introduction

It is well known that variation diminishing splines (VDS), introduced in the approximation theory during the eighties of the last century, have found many important applications in the field of integral-differential problems (see for example a survey in [1]).

In [2] Milovanovic and Kocic present an interesting application of the spline functional class in the field of computer graphics. Precisely, they propose an integral operator depending on a real parameter and based on variation dimininishing spline: the underlying properties of this new class of splines are particularly interesting in the field of free form curve modelling. We recall that a curve or surface is said to have a free form if it is possible to alter its shape by changing one or a few parameters with a priori knowledge of how this changing will affect the shape of the curve or surface.

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Following the idea suggested in [2], in the present paper we propose a particular class of bivariate tensor splines defined through an integral operator and depending on two parameters. It is named $\lambda - \tau$ integral VDS spline operator and is applied in the field of computer graphics, in order to obtain regularly behaving and pleasantly shaped surfaces, called B-spline integral models, with $\lambda - \tau$ shape parameters.

This paper is organized as follows: first we introduce the problem, going through the most significant results on univariate splines linked to an integral operator. In the second section we propose and analyse, as an extension of the univariate case, the bivariate case of the integral tensor splines operators. The third section is dedicated to the operator matrix expression which is used for the theoretical analysis and for the algorithm construction. The parameter optimization procedure is discussed in section four. An example illustrating the effectiveness of the proposed algorithm is presented in section five.

1. **Generation and properties of univariate integral parametric spline**

In this section we recall the basic concepts about VDS splines and the genesis of univariate integral parametric splines proposed in [2], to acquire the terminology and the motivations to build and study a new bivariate operator.

Given a set of vector points (*control points*) $\underline{P}_0$, $\underline{P}_1,\ldots,$ $\underline{P}_m$,(e.g. in a three-dimensional space) and a knots vector t:

$$0 = t_{-k} = ... = t_0 < t_1 < ... < t_{n-1} < t_n = ... = t_m = 1 \quad n = m - k,$$

the expression

$$(S_m P)(t) = \sum_{i=0}^{m} \underline{P}_i B_i^k(t) \quad 0 \leq t \leq 1 \tag{1}$$

is called a k-order variation diminishing spline operator (VDS operator) and generates a curve model called B-spline curve.

The basis function $B_i^k(t)$ $(i = 0, 1, \ldots m)$ are recursively defined as:

$$B_i^k(t) = \frac{t - t_{i-k}}{t_{i-1} - t_{i-k}} B_i^{k-1}(t) + \frac{t_i - t}{t_i - t_{i-k+1}} B_{i+1}^{k-1}(t)$$

$$\begin{aligned} B_i^0(t) &= 1 \quad t_i \le t \le t_{i+1} \\ B_i^0(t) &= 0 \quad otherwise \end{aligned}$$

In matrix form the VDS operator is:

$$(S_m P)(t) = \underline{b}_m(t)P \quad 0 \le t \le 1 \tag{2}$$

where:

$$\underline{b}_m = (B_0^k(t), B_1^k(t), ..., B_m^k(t)), \qquad P = (\underline{P}_0, \underline{P}_1, ..., \underline{P}_m)^T.$$

The authors in [2] proposed modifications to this class of splines by introducing a family of integral spline operators depending on a real parameter. We designate this new class as Univariate Integral $\lambda-$Variation Diminishing Splines.

Assuming that t_i is the value of the parameter corresponding to the given control point $\underline{P}_i$ we define:

$$\xi_i^k = \frac{t_{i-k+1} + ...t_i}{k}. \tag{3}$$

These points in the field of approximation are called Schonberg points [3]. We will call "correspondence points" such ξ_i^kvalues.

Let $x_j^i, (j = 1, 2, 3)$ be the generic component of vector $\underline{P}_i$ and $\varphi_j, (j = 1, 2, 3)$ the piecewise linear function interpolating points (ξ_i^k, x_j^i) and whose graphic is the control polygon.

The S_m operator on j-th component of P can then be expressed as:

$$(S_m P)_j = (S_m \varphi_j) = \sum_{i=0}^{m} \varphi_j(\xi_i^k) B_i^k(t), \quad j = 1, 2, 3$$

If we substitute $\varphi_j(\xi_i^k)$ by the integral mean:

$$\mu_i \varphi_j(t) = \frac{\int_{\xi_i^{k+1}}^{\xi_{i+1}^{k+1}} \varphi_j(u) du}{\xi_{i+1}^{k+1} - \xi_i^{k+1}} \tag{4}$$

we obtain the following operator T_m (integral VDS operator):

$$(S_m \mu_i \varphi_j) = (T_m \varphi_j) = (T_m P)_j \qquad (j = 1, 2, 3).$$

The T_m operator can be used to generate a new curve model and in matrix form it can be written as:

$$(T_m P)(t) = \underline{b}_m(t)(MP) \qquad 0 \leq t \leq 1 \tag{5}$$

Equation (5) shows that the integral spline can be regarded as the VDS operator produced by a new control points set Q, transforming in the global way the given set P. That is: $Q = MP$. Where matrix M has the following form:

$$M = \begin{bmatrix} \beta_0 & \gamma_0 & 0 & \dots & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \dots & 0 \\ 0 & \alpha_2 & \beta_2 & \dots & 0 \\ 0 & \dots & \dots & & \gamma_{m-1} \\ 0 & \dots & \dots & \beta_m & \gamma_m \end{bmatrix}$$

$$\begin{aligned} \alpha_0 &= 0, \alpha_i = \frac{(\delta_i^l)^2}{2\Delta_{i-1}^k \Delta_i^{k+1}}, i = 1, ...m; \\ \beta_i &= 1 - \alpha_i - \gamma_i, i = 1, ..., m \\ \gamma_i &= \frac{(\delta_i^r)^2}{2\Delta_i^k \Delta_i^{k+1}}, i = 1, ..., m, \gamma_m = 0 \end{aligned} \qquad \begin{aligned} \Delta_i^k &= \xi_{i+1}^k - \xi_i^k, \\ \delta_i^r &= \xi_{i+1}^{k+1} - \xi_i^k, \\ \delta_i^l &= \xi_i^k - \xi_i^{k+1}, \\ \xi_i^{k+1} &< \xi_i^k < \xi_{i+1}^{k+1} \end{aligned}$$

It follows that

$$(T_m P)(t) = (S_m Q)(t) \qquad 0 \leq t \leq 1$$

The obtained curve model is characterized by the following properties:

- it is invariant under affine transformations of the coordinate system;

- the whole curve lies inside the convex hull of the control polygon (the piecewise line whose vertices are the control points);

- it is uniquely determined by its control polygon and no two polygons produce the same curve;

- it crosses an arbitrary plane no more then does the control polygon ;

- it reproduces points and lines.

In [2] the authors introduce a shape parameter λ in the VDS integral operator. The integral mean expression in (4) is replaced by:

$$\mu_i^\lambda \varphi_j(t) = \frac{\int_{\zeta_i}^{\eta_i} \varphi_j(u)du}{\eta_i - \zeta_i} \tag{6}$$

where:

$$\begin{aligned} \zeta_i &= (1-\lambda)\xi_i^k + \lambda\xi_i^{k+1} \\ \eta_i &= (1-\lambda)\xi_i^k + \lambda\xi_{i+1}^{k+1} \end{aligned}$$

with $0 \leq t \leq 1$ and $0 \leq \lambda \leq 1$.

In matrix form this new operator can be written as:

$$(T_m^\lambda P)(t) = \underline{b}_m(t)(M^\lambda(\lambda)P) \qquad 0 \leq \lambda \leq 1 \tag{7}$$

where

$$M^\lambda(\lambda) = \begin{bmatrix} \beta_0^\lambda & \gamma_0^\lambda & 0 & \dots & 0 \\ \alpha_1^\lambda & \beta_1^\lambda & \gamma_1^\lambda & \dots & 0 \\ 0 & \alpha_2^\lambda & \beta_2^\lambda & \dots & 0 \\ 0 & \dots & \dots & & \gamma_{m-1}^\lambda \\ 0 & \dots & \dots & \beta_m^\lambda & \gamma_m^\lambda \end{bmatrix} \tag{8}$$

$$\begin{aligned} \alpha_i^\lambda &= \lambda\alpha_i \qquad i = 0, ..., m \\ \beta_i^\lambda &= 1 - \lambda(\alpha_i + \gamma_i), \qquad i = 0, ..., m \\ \gamma_i^\lambda &= \lambda\gamma_i \qquad i = 0, ..., m-1. \end{aligned}$$

$(T_m^\lambda P)$ is called integral spline VDS operator, with shape parameter. It can be shown that the λ parameter allows to control the global shape of the curve (whereas with the conventional spline only a local control can be achieved).

2. **The bivariate spline operator**

Now we extend the previously seen concepts of integral λ-VDS operator to the field of splines depending on two parameters (t and s).

This gives rise to a technique for describing surfaces in a three-dimensional space.

Let us organize our control points into $p+1$ sets of $m+1$ elements each, i.e.: $\underline{P}_{ij}$ $i = 0, 1, \ldots, p$ and $j = 0, 1, \ldots, m$ where $\underline{P}_{ij}$ is a three–dimensional vector. We call P the global set of control points.

We express the bivariate tensor VDS as:

$$(S_{mp}P)(t,s) = \sum_{i=0}^{p}\sum_{j=0}^{m} \underline{P}_{ij} C_{ij}^{kh}(t,s) \tag{9}$$

where the basis functions are obtained as a product of two univariate basis splines (of order k for the t parameter and h for the s parameter, respectively):

$$C_{ij}^{kh}(t,s) = B_i^k(t) B_j^h(s).$$

It can be easily seen from (9) that the bivariate tensor VDS is built on two classes of univariate VDS: a first one (*control curves*) controlled by the vector points $\underline{P}_{ij}$ and a second one (*swept curves*) controlled by points evaluated on first function class.

3. The matrix expression of the bivariate spline operator

We suggest to express the l-th component of bivariate tensor VDS, in matrix form, as follows:

$$(S_{mp}P)(t,s)_l = \underline{b}^{km}(t) \begin{bmatrix} p_l^{00} & \ldots & p_l^{0p} \\ \ldots & \ldots & \ldots \\ p_l^{m0} & \ldots & p_l^{mp} \end{bmatrix} (\underline{b}^{hp}(s))^T$$

i.e.:

$$(S_{mp}(t,s))_l = \underline{b}^{km} P_l (\underline{b}^{hp})^T \tag{10}$$

where:

$$\underline{b}^{wr} = (B_0^w(t), B_1^w(t), \ldots, B_r^w(t))$$

and p_l^{ij} is the l-th component of vector $\underline{P}_{ij}$.

By exploiting the separability of the tensor product basis functions, it is possible to extend the formalism introduced in the univariate case to the bivariate case.

Therefore: the control points for the *control spline* function are modified by the same matrix used in the univariate case, then the points on these splines (control points for *swept splines*) are modified by another similar matrix.

We get:

$$(T_{mp}^{\lambda\tau}P)_l(t,s) = \underline{b}^{km}(t)M^{\tau}(\tau)P_l M^{\lambda}(\lambda)(\underline{b}^{hp}(s))^T \tag{11}$$

where $M^{\tau}(\tau)$ is a $(m+1)$-order square matrix, depending on the knots of an h-order B-spline, having the same expression as for univariate splines.

Similarly, $M^{\lambda}(\lambda)$ is a $(p+1)$-order square matrix, depending on the knots of an h-order B-spline.

Theorem 1. *Let us consider the nonlinear operator(11). An algebraic and sintetic expression of it is the following:* $(1-\tau)(1-\lambda)(S_{mp}^{\lambda\tau}P)_l + \tau(1-\lambda)(S_{mp}^{\lambda\tau}Q^{\tau})_l + \lambda(1-\tau)(S_{mp}^{\lambda\tau}Q^{\lambda})_l + \lambda\tau(S_{mp}^{\lambda\tau}Q^{\lambda\tau})_l$, *where:* $Q^{\tau} = M^{\tau}(1)P$, $Q^{\lambda} = PM^{\lambda}(1)$, $Q^{\lambda\tau} = M^{\tau}(1)PM^{\lambda}(1)$

Proof. The proof is based on the following relationship: $M^{\alpha}(\alpha) = (1-\alpha)I + \alpha M(1)$. Substituing it into (11) we get: $\underline{b}^{km}((1-\tau)I^{(m+1)} + \tau M^{\tau}(1))P_l((1-\lambda)I^{(p+1)} + \lambda M^{\lambda}(1))(\underline{b}^{hp}(s))^T$ through some algebraic steps the thesis follows. □

4. **Parameters optimization**

In this section we will deal with the problem of finding optimal values for λ and τ.

The aim is to obtain the "best" reconstruction of a surface starting from a cloud of measured points.

The first possibility to find optimal λ and τ parameters is to minimize a quadratic functional expressing the global (Euclidean) distance of the given data points from the correspondence points on the reconstructed surface. The functional has the following expression:

$$F(\lambda,\tau)=\sum_{l=1}^{3}\sum_{j=0}^{m}\sum_{i=0}^{p}\delta^2(\underline{P}_l^{ij},((T_{mp}^{\lambda\tau}P)(\xi_i^k,\xi_j^h))_l)$$

This solution gives the most precise representation of a given set of points, but is very sensitive to digitizing errors.

A second approach consists in minimizing the energy-functional:

$$F(\lambda,\tau)=\sum_{l=1}^{3}\int_D(\frac{\partial^2}{\partial t^2}(T_{mp}^{\lambda\tau}P)(\xi_i^k,\xi_j^h))_l+\frac{\partial^2}{\partial s^2}(T_{mp}^{\lambda\tau}P)(\xi_i^k,\xi_j^h))_l d\lambda d\tau$$

The D domain corresponds to the whole variation of the t and s parameters, relevant to the considered surface.

This algorithm gives satisfactory results as far as the surface smoothness is involved.

5. **Test example**

The following example higlights the noise sensitivity of the computed surface. The saddle surface whose equation is $z = x^2 - y^2$ (hyperbolic paraboloid) has been used for testing.

The left part Figure 1 shows the “measured” points on the surface; on the right of the same figure the surface reconstructed using usual splines function is shown.

The left part of Figure 2 represents the surface reconstructed using the minimization of the distance-functional while on the right part the reconstructed surface by means of minimizing the energy functional is shown. The first surface, which is satisfactory as algorithm test, furthemore still presents some irregularity, on the contrary the second one looks very smooth.

6. **Conclusion**

We have proposed a non linear bivariate operator based on an integral parametric spline family. By this operator it is possible to obtain a smooth surface, without modifying each single control point; such surfaces exhibit interesting properties as far as engineering applications are involved. The next activities we intend to carry out

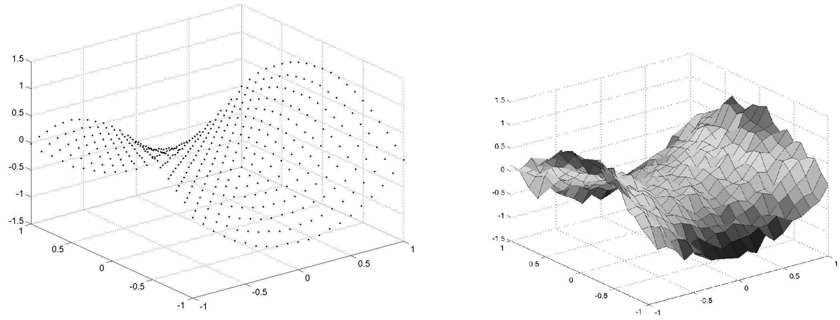

FIGURE 1. Left: "Measured" points. Right: Reconstructed surface using conventional splines functions.

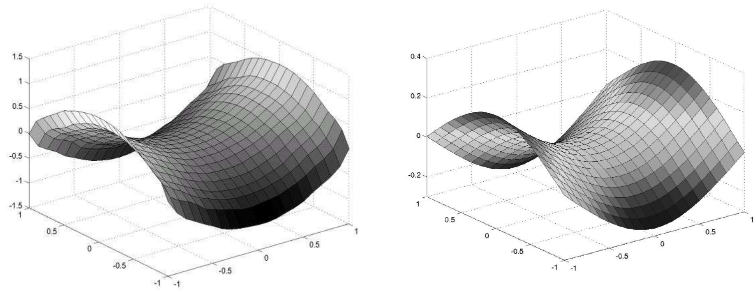

FIGURE 2. Left: Reconstructed surface obtained minimizing the distance-functional. Right: Reconstructed surface obtained minimizing the energy-functional.

are: the theoretical investigation of geometrical properties, to acquire a wider record of application cases and finally to study other functionals to obtain the optimal values of thye shape parameters.

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