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SMOOTH DEPENDENCE OF SOLUTION ON PARAMETERS FOR THE VOLTERRA-FREDHOLM INTEGRAL EQUATION

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Abstract. In this paper we will give conditions that ensures the differentiability with respect to parameters of the solution of Volterra-Fredholm nonlinear integral equation.

1. **Introduction**

In the present paper consider the nonlinear integral equation of Volterra-Fredholm type:

$$u(x,t) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u(y,s))dyds \tag{1}$$

$\forall t \in [0,c]$, $\forall x \in [\alpha,\beta]$, where $[a,b] \subset [\alpha,\beta]$.

Applying fiber Picard operators theory, we will prove the differentiability of the solution of (1) with respect to a and b.

2. **Fiber Picard operators**

Let (X,d) be a metric space and $A : X \to X$ an operator. In this paper we will use the following notations:

$$F_A := \{x \in X : A(x) = x\};$$

$$A^0 := 1_X\,,\ A^{n+1} := A \circ A^n \ \forall n \in \mathbb{N}.$$

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Definition 2.1. *(I. A. Rus [1]) The operator A is said to be:*
(i) ***weakly Picard operator (wPo)*** *if $\forall x_0 \in X$ $A^n(x_0) \to x_0^*$, and the limit x_0^* is a fixed point of A, which may depend on x_0.*
(ii) ***Picard operator (Po)*** *if $F_A = \{x^*\}$ and $\forall x_0 \in X$ $A^n(x_0) \to x^*$.*

In the next section we need the following result:

Theorem 2.1. *(Fiber Contraction Principle, I. A. Rus [3]) Let (X,d), (Y,ρ) be two metric spaces and $B : X \to X,\ C : X \times Y \to Y$ two operators such that:*
(i) (Y,ρ) is complete;
(ii) B is a Picard operator, $F_B = \{x^*\}$;
(iii) $C(\cdot, y) : X \to Y$ is continuous $\forall y \in Y$;
(iv) $\exists a \in]0,1[$ such that the operator $C(x,\cdot) : Y \to Y$ is an a-contraction for all $x \in X$; let y^* be the unique fixed point of $C(x^*,\cdot)$.
Then

$$A : X \times Y \to X \times Y, \quad A(x,y) := (B(x), C(x,y))$$

is a Picard operator and $F_A = \{(x^*, y^*)\}$.

This theorem is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters. For such results see [6], [3], [2], [4], [5], etc.

3. Main result

Theorem 3.1. *Consider the equation (1) in the next conditions:*
(i) $f \in C([a,b] \times [0,c])$ and $K \in C([a,b] \times [0,c] \times [a,b] \times [0,c] \times \mathbb{R})$;
(ii) there exists $L_K > 0$ such that:

$$|K(x,t,y,s,u) - K(x,t,y,s,v)| \le L_K |u - v| \tag{2}$$

$\forall (x,t,y,s) \in [\alpha,\beta] \times [0,c] \times [\alpha,\beta] \times [0,c]$, $\forall u,v \in \mathbb{R}$.
Then:
a) for all $a < b \in [\alpha,\beta]$, the equation (1) has in $C([\alpha,\beta] \times [0,c])$ a unique solution

$u^*(\cdot,\cdot,a,b)$.

b) for all $u_0 \in C([\alpha,\beta]\times[0,c])$*, the sequence* $(u_n)_{n\geq 0}$ *defined by:*

$$u_n(x,t,a,b) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u_{n-1}(y,s,a,b))dyds$$

converges uniformly to u^**,* $\forall (x,t,a,b) \in [\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta]$.

c) The function u^**,* $(x,t,a,b) \mapsto u^*(x,t,a,b)$ *is continuous:* $u^* \in C([\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta])$*;*

d) If $K(x,t,y,s,\cdot) \in C^1(\mathbb{R})$*,* $\forall (x,t,y,s) \in [\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[0,c]$*, then* $u^*(x,t,\cdot,\cdot) \in C^1([\alpha,\beta]\times[\alpha,\beta])$*,* $\forall (x,t) \in [\alpha,\beta]\times[0,c]$.

Proof. Let the space $C([a,b]\times[0,c],\mathbb{R})$ be endowed with a suitable norm,

$$\|u\|_{BC} := \sup\{\|u(x,t)\|\, e^{-\tau t} : x \in [a,b], t \in [0,c]\}, \quad \tau > 0 \tag{3}$$

Let $X := C([\alpha,\beta]\times[0,c]\times[\alpha,\beta]\times[\alpha,\beta])$. We consider the operator $B : X \to X$ defined by:

$$B(u)(x,t,a,b) := f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u(y,s,a,b))dyds$$

From (ii), applying the Contraction Principle, it follows that B is a contraction, so we have a), b) and c).

For all $a < b \in [\alpha,\beta]$, there is a unique solution $u^*(\cdot,\cdot,a,b) \in C([\alpha,\beta]\times[0,c])$, so we have:

$$u^*(x,t,a,b) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u^*(y,s,a,b))dyds \tag{4}$$

Let us prove that $\dfrac{\partial u^*(x,t,a,b)}{\partial a}$ and $\dfrac{\partial u^*(x,t,a,b)}{\partial b}$ exists and they are continuous.

1. Supposing that $\dfrac{\partial u^*(x,t,a,b)}{\partial a}$ exists, from (4) we obtain:

$$\frac{\partial u^*(x,t,a,b)}{\partial a} = -\int_0^t K(x,t,a,s,u^*(a,s,a,b))ds +$$

$$+\int_0^t \int_a^b \frac{\partial K(x,t,y,s,u^*(y,s,a,b))}{\partial u} \cdot \frac{\partial u^*(y,s,a,b)}{\partial a} dyds$$

This relationship suggest us to consider the next operator:

$C : X \times X \to X$, defined by:

$$C(u,v)(x,t,a,b) := -\int_0^t K(x,t,a,s,u(a,s,a,b))ds +$$

$$+\int_0^t \int_a^b \frac{\partial K(x,t,y,s,u(y,s,a,b))}{\partial a} \cdot v(y,s,a,b)dyds$$

Let u^* be the unique fixed point of B. The operator $C(u,\cdot)$ is a contraction $\forall u \in X$ and let v^* be the unique fixed point of $C(u^*,\cdot))$.

If we define the operator $A : X \times X \to X \times X$,

$$A(u,v)(x,t,a,b) := (B(u)(x,t,a,b), C(u,v)(x,t,a,b)),$$

then the conditions of the Theorem 2.1 are fulfilled. It follows that A is a Picard operator and $F_A = \{(u^*,v^*)\}$.

Consider the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ defined by:

$$u_n(x,t,a,b) := B(u_{n-1}(x,t,a,b)) =$$

$$= f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u_{n-1}(y,s,a,b))dyds \quad \forall n \geq 1$$

$$v_n(x,t,a,b) := C(u_{n-1}(x,t,a,b), v_{n-1}(x,t,a,b)) =$$

$$= -\int_0^t K(x,t,a,s,u_{n-1}(a,s,a,b))ds +$$

$$+\int_0^t \int_a^b \frac{\partial K(x,t,y,s,u_{n-1}(y,s,a,b))}{\partial u} \cdot v_{n-1}(y,s,a,b)dyds \quad \forall n \geq 1$$

We have:

$$u_n \rightrightarrows u^* \ (n \to \infty), \quad v_n \rightrightarrows v^* \ (n \to \infty) \tag{5}$$

uniformly for $(x,t,a,b) \in [\alpha,\beta] \times [0,c] \times [\alpha,\beta] \times [\alpha,\beta]$.

We take $u_0 = v_0 := 0$, so $v_1 = \dfrac{\partial u_1}{\partial a}$.

By induction we can prove that $v_n = \dfrac{\partial u_n}{\partial a}\ \forall n$ and from (5) results:

$$\frac{\partial u_n}{\partial\ a} \rightrightarrows v^* \ (n \to \infty)$$

Using a Weierstrass theorem, it follows that $\frac{\partial u^*}{\partial\, a}$ exists and

$$\frac{\partial u^*(x,t,a,b)}{\partial a} = v^*(x,t,a,b).$$

2. By a similar way, we can prove the existence and the continuity of $\frac{\partial u^*}{\partial\, b}$.□

Remark 3.1. *We can also consider the following integral equation of Volterra-Fredholm type:*

$$u(x,t) = f(x,t) + \int_0^t \int_a^b K(x,t,y,s,u(y,s),\lambda)dyds \tag{6}$$

$\forall t \in [0,c]$, $\forall x \in [a,b]$, *where* $\lambda \in \mathbb{R}$ *and we can prove the differentiability of the solution with respect to the parameter* λ.

This case will be presented elsewhere.

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