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TWO- AND THREE-DIMENSIONAL INVERSE PROBLEM OF DYNAMICS

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Abstract. For a given a monoparametric family of curves $f(x, y) = c$, we present the partial differential equations satisfied by the potentials $V = V(x, y)$ under whose action a particle of unit mass can describe the curves of the family. Szebehely's equation depends on the total energy of the particle, while Bozis' one relates merely the potential and the given family. Therefore the last one is also adequate for the direct problem of dynamics. A similar program is accomplished for a two-parametric spatial family of curves $\varphi(x, y, z) = c_1$, $\psi(x, y, z) = c_2$ and potentials $\mathcal{V} = \mathcal{V}(x, y, z)$.

1. Introduction

The first result concerning the inverse problem of dynamics is due to Newton [24], who presented the form of the gravitational potential on the basis of Kepler's laws. Kepler has had at his disposal the very accurate tables of observations made by Tycho Brache (whose assistant he was in Prague); these observations allowed him to discover that the orbit of Mars is an ellipse and to formulate the three laws of planetary motion.

Later on, Bertrand [7] showed that Kepler's first law suffices to derive the Newtonian universal force; Dainelli [18] obtained the expressions of general force fields producing given planar or spatial families of curves.

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The two-dimensional problem, this time for conservative systems, has renewed the interest in the inverse problem of dynamics by means of Szebehely's [29] partial differential equation. This equation relates the potential to the given monoparametric family of curves and to the total energy. Puel [26] derived a Szebehely-type equation which is independent of the coordinate system. Another basic result for the two-dimensional inverse problem is the energy-free partial differential equation obtained by Bozis [9] from Szebehely's equation, and later derived directly by Anisiu [3].

The conservative three-dimensional problem was considered by Érdi [19] for a monoparametric family of orbits, and then for two-parametric families by Váradi and Érdi [30]. Puel [25] used the least action principle of Maupertuis to obtain the equations satisfied by the potential in the two- and three-dimensional inverse problem of dynamics. The existence of such a potential and its relation with the energy in the three-dimensional case was subject to further papers, as those of Gonzales-Gascon et al [21], Bozis and Nakhla [15] and Shorokhov [28]. Puel [27] obtained the intrinsic equations of the three-dimensional inverse problem, using the Frenet reference frame. A review of the basic results in the inverse problem of dynamics, including the three-dimensional ones, can be found in [10].

2. **The planar inverse problem of dynamics**

We consider the following version of the inverse problem for one material point of unit mass, moving in the xy inertial Cartesian plane. Given a family of curves

$$f\left(x,y\right)=c \tag{1}$$

with f of C^3-class (continuous and with continuous derivatives up to third order on a domain of the plane), find the potentials $V(x,y)$ under whose action, for appropriate initial conditions, the particle will describe the curves of that family. The equations of motion are

$$\ddot{x}=-V_x \qquad \ddot{y}=-V_y, \tag{2}$$

where the dots denote derivatives with respect to the time t and the subscripts partial derivatives. By making use of the energy integral, Szebehely [29] proved that the potential V is a solution of the first order partial differential equation

$$f_x V_x + f_y V_y + \frac{2\,(V - E(f))}{f_x^2 + f_y^2}\left(f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_{yy} f_x^2\right) = 0, \tag{3}$$

where $E(f)$ denotes the total energy, which is constant on each curve of the family (1). Bozis [8] wrote Szebehely's equation in the simpler form

$$V_x + \gamma V_y + \frac{2\Gamma\,(E(f) - V)}{1 + \gamma^2} = 0, \tag{4}$$

making use of the functions

$$\gamma = \frac{f_y}{f_x} \quad \text{and} \quad \Gamma = \gamma\gamma_x - \gamma_y \tag{5}$$

related to the geometry of the family (γ representing the slope and Γ being proportional to the curvature). By eliminating the energy from (4) (using the fact that $E_y/E_x = f_y/f_x$) Bozis [9] obtained the energy-free equation of second order

$$-V_{xx} + \kappa V_{xy} + V_{yy} = \lambda V_x + \mu V_y, \tag{6}$$

where

$$\kappa = \frac{1}{\gamma} - \gamma, \ \lambda = \frac{\Gamma_y - \gamma\Gamma_x}{\gamma\Gamma}, \ \mu = \lambda\gamma + \frac{3\Gamma}{\gamma}. \tag{7}$$

The basic equations (4) and (6) of the planar inverse problem of dynamics present the connection between geometry and dynamics. Their derivation and other related results are exposed in [10], [2], [1], [3].

Szebehely obtained the first order equation intending to determine the potential of the earth by means of satellite observations, while Bozis used equation (6) to check if a given family of orbits may be generated in the plane of symmetry outside a material concentration.

2.1. **Basic tools.** Let us consider a particle whose motion is described by equations (2), where V is of C^2-class on a domain of the xy plane. We shall use a procedure exposed by Anisiu [3], related to that followed by Kasner [22] while he has obtained the differential equation of the trajectories corresponding to a general (not necessarily conservative) force field. By differentiating (1) with respect to t we get $f_x\dot{x}+f_y\dot{y}=0$, or, using notation (5),

$$\gamma = -\frac{\dot{x}}{\dot{y}}. \tag{8}$$

By differentiating (8) we get

$$-\Gamma = \frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{y}^3}. \tag{9}$$

Inserting in (9) $\ddot{x}$ and $\ddot{y}$ from (2), and $\dot{x}$ from (8) we obtain

$$\Gamma\dot{y}^2 = -\left(V_x+\gamma V_y\right).$$

The function Γ is related to the curvature K of the family (1) by $K = |\Gamma| / \left(\gamma^2+1\right)^{3/2}$. It follows that $\Gamma = 0$ if and only if the family (1) contains only straight lines. In this case, which was studied in [11], we have by necessity

$$V_x+\gamma V_y = 0, \tag{10}$$

which represents Szebehely's equation for this special case. The straight lines are traced with arbitrary energy.

Let us consider now a general family (1) with $\Gamma \neq 0$. In this case we have

$$\dot{y}^2 = -\frac{V_x+\gamma V_y}{\Gamma}. \tag{11}$$

We differentiate (9), divide both members by $\dot{y}$ and get

$$\gamma\Gamma_x - \Gamma_y = \frac{\dot{y}\left(\dot{x}\dddot{y}-\dot{y}\dddot{x}\right)-3\ddot{y}\left(\dot{x}\ddot{y}-\dot{y}\ddot{x}\right)}{\dot{y}^5}. \tag{12}$$

We remark that (8), (9) and (12) express the relations between the geometry of the family of curves (1) and the kinematics derivatives.

Two additional equations are obtained by differentiating equations (2) with respect to t, namely

$$\begin{aligned} \dddot{x} &= -\left(V_{xx}\dot{x} + V_{xy}\dot{y}\right) \\ \dddot{y} &= -\left(V_{xy}\dot{x} + V_{yy}\dot{y}\right). \end{aligned} \tag{13}$$

Now we eliminate the derivatives $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \dddot{x}, \dddot{y}$ between the seven relations in (2), (8), (11), (12) and (13), and get the partial differential equation

$$\Gamma\left(-\gamma V_{xx} + V_{xy} - \gamma^2 V_{xy} + \gamma V_{yy}\right) = -\left(V_x + \gamma V_y\right)\left(\gamma\Gamma_x - \Gamma_y\right) + 3V_y\Gamma^2. \tag{14}$$

We divide both members of (14) by $\gamma\Gamma$ and obtain Bozis' equation (6), with λ and μ given in (7).

A straightforward calculation shows that equation (6) can be written as

$$\gamma W_x - W_y = 0, \tag{15}$$

where

$$W = V - \frac{1+\gamma^2}{2\Gamma}\left(V_x + \gamma V_y\right). \tag{16}$$

Equation (15) has the general solution $W = E(f)$, where E denotes an arbitrary function. It follows that

$$V - \frac{1+\gamma^2}{2\Gamma}\left(V_x + \gamma V_y\right) = E(f). \tag{17}$$

In view of relations (2), (8) and (9) we obtain

$$V + \frac{\dot{x}^2 + \dot{y}^2}{2} = E(f), \tag{18}$$

which means that $E(f)$ represents the total energy, constant on each curve of the family (1). Therefore equation (17), obtained this time from Bozis' equation, is in fact Szebehely's equation. From (18) we obtain $E(f) - V \geq 0$, and from (17) it follows that only the curves of the family (1) or parts of them which are situated in the plane region

$$\frac{V_x + \gamma V_y}{\Gamma} \leq 0 \tag{19}$$

can be described by the unit mass particle. Inequality (19) was obtained by Bozis and Ichtiaroglou [12].

Remark 1. *Bozis* [10] *arranged equation (6) in a form adequate for the direct problem of dynamics, namely*

$$\gamma^2\gamma_{xx} - 2\gamma\gamma_{xy} + \gamma_{yy} = h, \tag{20}$$

where

$$h = \frac{\gamma\gamma_x - \gamma_y}{V_y\gamma + V_x}\left(-\gamma_x V_x + \left(2\gamma\gamma_x - 3\gamma_y\right)V_y + \gamma\left(V_{xx} - V_{yy}\right) + (\gamma^2 - 1)V_{xy}\right). \tag{21}$$

Relations (20)-(21) have been used to find families of curves satisfying auxiliary conditions, supposing that a potential is given, in [16], [17], [6].

2.2. **Examples.**

Example 2. *From the class of Hénon-Heiles potentials*

$$V(x,y) = ax^2 + by^2 + cx^2y + dy^3 \tag{22}$$

with $a,b,c,d \in \mathbb{R},\ a,b > 0$, *Anisiu and Pal* [5] *looked for those compatible with the family of polytropic curves* $f(x,y) = x^{-p}y$, *where* $p \in \mathbb{Z} \setminus \{0,1\}$. *The potential*

$$V_1(x,y) = a\left(x^2 + 16y^2\right) + c\left(x^2 + (16/3)y^2\right)y$$

was found to generate the family $f_1(x,y) = x^{-4}y$ *in the region described by* $y(cx^2 + 8cy^2 + 24ay) \leq 0$, *with the energy* $E_1(f_1) = -c/(24f_1)$. *Another potential is*

$$V_2(x,y) = a\left(x^2 + 4y^2\right) + dy^3,$$

which produces the family $f_2(x,y) = x^2y$ *in the region* $dy + 4a \leq 0$, *with the energy* $E_2(f_2) = -df_2/4$.

It was shown in [11] *that no potential of the form (22) allows for families of straight lines.*

Example 3. *For the family* $f = y - 1/x^2$, *the potential*

$$V(x,y) = 8y^2 + 4x^2y - x^8 - 6x^2$$

was found in [17]. *The particle describes the curves of the given family in the region* $y \leq x^4 + 1/(2x^2)$ *with the energy* $E(f) = 8f^2$.

3. **The three-dimensional inverse problem**

We consider the three-dimensional family of curves

$$\varphi\,(x,y,z)=c_1,\quad \psi\,(x,y,z)=c_2. \tag{23}$$

with φ,ψ of C^3-class and with

$$\begin{vmatrix} \varphi_y & \varphi_z \\ \psi_y & \psi_z \end{vmatrix} \neq 0. \tag{24}$$

We can suppose that any other determinant (containing derivatives with respect to x and y, or to x and z) is different from zero, and proceed accordingly.

We deal with the following version of the inverse problem: find the potentials $\mathcal{V}\,(x,y,z)$ under whose action, for appropriate initial conditions, a material point of unit mass, whose motion is described by

$$\ddot{x}=-\mathcal{V}_x \quad \ddot{y}=-\mathcal{V}_y \quad \ddot{z}=-\mathcal{V}_z, \tag{25}$$

will trace the curves of the family (23). The partial differential equations satisfied by $\mathcal{V}$ will be derived as in [4], where the geometrical methods used by Kasner [23] were adapted to this problem.

3.1. **Basic tools.** In order to obtain the equations satisfied by $\mathcal{V}$, we differentiate both sides of equations (23) with respect to t, and get

$$\frac{\dot{y}}{\dot{x}}=\alpha,\quad \frac{\dot{z}}{\dot{x}}=\beta, \tag{26}$$

where

$$\alpha=\frac{\varphi_z\psi_x-\varphi_x\psi_z}{\varphi_y\psi_z-\varphi_z\psi_y},\quad \beta=\frac{\varphi_x\psi_y-\varphi_y\psi_x}{\varphi_y\psi_z-\varphi_z\psi_y}. \tag{27}$$

We remark that at least one of the functions α and β, say α, is not identically null (otherwise condition (24) fails to be fulfilled).

The notation (27) was introduced by Bozis and Kotoulas [13], where it was emphasized that the family (23) leads to a unique pair α, β and, conversely, the pair α, β determines uniquely the family (23).

We differentiate both relations in (26) and get

$$\frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{\dot{x}^3}=A, \quad \frac{\dot{x}\ddot{z}-\ddot{x}\dot{z}}{\dot{x}^3}=B, \tag{28}$$

where

$$A=\alpha_x+\alpha\alpha_y+\beta\alpha_z, \quad B=\beta_x+\alpha\beta_y+\beta\beta_z. \tag{29}$$

Using (26) and equations (25), we obtain from (28)

$$\frac{\alpha\mathcal{V}_x-\mathcal{V}_y}{\dot{x}^2}=A, \quad \frac{\beta\mathcal{V}_x-\mathcal{V}_z}{\dot{x}^2}=B. \tag{30}$$

We have to analyze the special case when $A=B=0$. It is obvious that, in view of relation (28), it follows that also $\dot{y}\ddot{z}-\ddot{y}\dot{z}=0$, hence the curvature $K=|\dot{\overline{r}}\times\ddot{\overline{r}}|/|\dot{\overline{r}}|^3$ of each member of the family (23) vanishes. We have denoted by $\overline{r}=x\,(t)\,\overline{i}+y\,(t)\,\overline{j}+z\,(t)\,\overline{k}$, where $\overline{i}$, $\overline{j}$, $\overline{k}$ are unit vectors along the axes Ox, Oy, Oz.

It follows that we have $A=B=0$ if and only if the family (23) consists of straight lines. This case was analyzed in detail in [13]. Relations (30) give rise to two linear partial differential equations to be necessarily satisfied by $\mathcal{V}$, namely

$$\alpha\mathcal{V}_x-\mathcal{V}_y=0, \quad \beta\mathcal{V}_x-\mathcal{V}_z=0. \tag{31}$$

These equations will admit of a solution only if α and β satisfy, besides the two equations obtained from (29) for $A=B=0$, a supplementary equation (see [20])

$$\alpha\beta_x-\beta\alpha_x=\beta_y-\alpha_z. \tag{32}$$

So, generally, the inverse problem is not expected to have a solution for arbitrary families of straight lines.

Let us consider now $A\neq 0$ and $B\neq 0$. By eliminating $\dot{x}^2$ between the two relations in (30) we obtain a first necessary condition to be satisfied by $\mathcal{V}$,

$$\frac{\alpha\mathcal{V}_x-\mathcal{V}_y}{A}=\frac{\beta\mathcal{V}_x-\mathcal{V}_z}{B}, \tag{33}$$

where α,β from (27) and A,B from (29) depend on the derivatives of φ and ψ up to the second order. Because of $\dot{x}^2\geq 0$, it follows that the motion is possible only in the region determined by

$$\frac{\alpha\mathcal{V}_x-\mathcal{V}_y}{A}\geq 0. \tag{34}$$

Differentiating both members of the equality $\dot{x}^2 = (\alpha\mathcal{V}_x - \mathcal{V}_y)/A$ with respect to t and replacing $\ddot{x}$ from the first equation in (25), respectively $\dot{y}/\dot{x}$ and $\dot{z}/\dot{x}$ from (26), we obtain a second differential relation to be satisfied by $\mathcal{V}$

$$-\mathcal{V}_{xx} + k\mathcal{V}_{xy} + \mathcal{V}_{yy} + p\mathcal{V}_{yz} + q\mathcal{V}_{xz} = l\mathcal{V}_x + m\mathcal{V}_y, \tag{35}$$

where

$$\begin{gathered} k = \frac{1}{\alpha} - \alpha,\ p = \frac{\beta}{\alpha},\ q = -\beta \\ l = \frac{3A}{\alpha} - \alpha m,\ m = \frac{A_x + \alpha A_y + \beta A_z}{\alpha A}. \end{gathered} \tag{36}$$

Summarizing the above reasoning, we assert that a potential which produces as orbits the curves of the family (23) satisfies by necessity the two differential relations (33) and (35), the motion of the particle being possible in the region determined by inequality (34). We remark that equation (35) is of second order in $\mathcal{V}$ and does not involve the energy (constant on each curve of the family), hence it is the corresponding for the three-dimensional case of Bozis' equation (6) satisfied by planar potentials.

In the following we shall derive the equation from which the total energy can be expressed. Denoting by

$$\mathcal{W} = \left(1 + \alpha^2 + \beta^2\right)\frac{\alpha\mathcal{V}_x - \mathcal{V}_y}{2A} + \mathcal{V}, \tag{37}$$

one can check by direct calculation that (35) is equivalent to

$$\mathcal{W}_x + \alpha\mathcal{W}_y + \beta\mathcal{W}_z = 0. \tag{38}$$

The characteristic system for (38) is

$$\frac{dx}{\varphi_y\psi_z - \varphi_z\psi_y} = \frac{dy}{\psi_x\varphi_z - \varphi_x\psi_z} = \frac{dz}{\varphi_x\psi_y - \varphi_y\psi_x}$$

and one obtains easily that $\varphi_x dx + \varphi_y dy + \varphi_z dz = 0$ and $\psi_x dx + \psi_y dy + \psi_z dz = 0$. It follows that $\varphi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$ are integrals, hence the general solution of (38) is $\mathcal{W} = \mathcal{E}(\varphi, \psi)$ with $\mathcal{E}$ an arbitrary function.

In view of relations (26) and (30), we get from (37) that

$$\mathcal{E}(\varphi, \psi) = \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)/2 + \mathcal{V}, \tag{39}$$

i.e. $\mathcal{W} = \mathcal{E}(\varphi, \psi)$ is the total energy, constant on each curve of the family (23). It follows that the equation

$$\mathcal{E}(\varphi, \psi) = \left(1 + \alpha^2 + \beta^2\right) \frac{\alpha \mathcal{V}_x - \mathcal{V}_y}{2A} + \mathcal{V}, \tag{40}$$

which was derived by Váradi and Érdi [30] using the energy integral (and which corresponds to Szebehely's planar equation), can be obtained as a consequence of the second order partial differential equation (35).

The two equations (33) and (35) for a single unknown function $\mathcal{V}$ will not have always a solution; compatibility conditions are to be checked. The advantage of this formulation consists in the fact that it is free of energy.

Remark 4. *Equations (33) and (35) are suitable for the direct problem of dynamics: given a three-dimensional potential, find families of curves of the form (23) generated by it. We can rearrange the mentioned equations and obtain a linear partial differential equation of first order in* α *and* β

$$(\mathcal{V}_x \beta - \mathcal{V}_z)(\alpha_x + \alpha \alpha_y + \beta \alpha_z) - (\mathcal{V}_x \alpha - \mathcal{V}_y)(\beta_x + \alpha \beta_y + \beta \beta_z) = 0, \tag{41}$$

and a nonlinear one of second order

$$\begin{aligned} &\alpha_{xx} + \alpha^2 \alpha_{yy} + \beta^2 \alpha_{zz} + 2\alpha \alpha_{xy} + 2\beta \alpha_{xz} + 2\alpha\beta \alpha_{yz} = \\ &\frac{A}{\mathcal{V}_x \alpha - \mathcal{V}_y} \cdot \left(3\mathcal{V}_x \alpha_x + (2\mathcal{V}_x \alpha + \mathcal{V}_y)\alpha_y + (2\mathcal{V}_x \beta + \mathcal{V}_z)\alpha_z \right. \\ &\left. + \mathcal{V}_{xx}\alpha - \mathcal{V}_{xy}\left(1 - \alpha^2\right) - \mathcal{V}_{yy}\alpha - \mathcal{V}_{yz}\beta + \mathcal{V}_{xz}\alpha\beta\right). \end{aligned} \tag{42}$$

If $B = 0$ and $A \neq 0$, we still have inequality (34); instead of (33), the relation $\beta \mathcal{V}_x - \mathcal{V}_z = 0$ holds, beside the second order partial differential equation (35).

If $A = 0$ and $B \neq 0$, the inequality to be satisfied is $(\beta \mathcal{V}_x - \mathcal{V}_z)/B \geq 0$, and (33) is replaced by $\alpha \mathcal{V}_x - \mathcal{V}_y = 0$. Starting with $\dot{x}^2 = (\beta \mathcal{V}_x - \mathcal{V}_z)/B$, we follow the steps from the case when both A and B were different from zero and obtain instead of (35)

$$-\mathcal{V}_{xx} + \tilde{k}\mathcal{V}_{xz} + \mathcal{V}_{zz} + \tilde{p}\mathcal{V}_{yz} + \tilde{q}\mathcal{V}_{xy} = \tilde{l}\mathcal{V}_x + \tilde{m}\mathcal{V}_z, \tag{43}$$

where

$$\tilde{k} = \frac{1}{\beta} - \beta,\ \tilde{p} = \frac{\alpha}{\beta},\ \tilde{q} = -\alpha$$
$$\tilde{l} = \frac{3B}{\beta} - \beta\tilde{m},\ \tilde{m} = \frac{B_x + \alpha B_y + \beta B_z}{\beta B}. \tag{44}$$

3.2. **Examples.**

Example 5. *The two-parametric family of straight lines*

$$\frac{y}{x} = c_1, \quad \frac{z}{x} = c_2$$

was found in [13] *to be compatible with the (central) potential*

$$\mathcal{V}(x, y, z) = F(x^2 + y^2 + z^2),$$

where F is an arbitrary function of its argument.

Shorokhov [28] *presented a family of straight lines*

$$\frac{x}{y} = c_1, \quad y + z = c_2$$

which cannot be described by a particle under the action of any potential. This family has $\alpha = y/x$ and $\beta = -y/x$, hence condition (31) does not hold.

Example 6. *The family of curves*

$$\frac{z}{x} = c_1, \quad x^2 + y^2 = c_2$$

was considered in [30] *and* [15]*. It can be traced all over the space under the action of the potential*

$$\mathcal{V}(x, y, z) = (x^2 + y^2 + z^2)/2,$$

with the energy $\mathcal{E}(\varphi, \psi) = \psi(\varphi^2 + 2)/2$. This example illustrates the case $A \neq 0$, $B = 0$.

Example 7. *For the family of curves*

$$x^2 + y^2 = c_1, \quad \frac{x^2 - y^2}{z} = c_2$$

one has $A \neq 0$ and $B \neq 0$. The potential

$$\mathcal{V}(x, y, z) = x^2 + y^2 + 4z^2$$

given in [14] *produces the given family with the energy* $\mathcal{E}(\varphi,\psi)=2\varphi\left(2\varphi+\psi^2\right)/\psi^2$.

4. Conclusions

The energy-free equations have a basic role in the inverse problem of dynamics. When we have no a priori information on the energy of the given family, it is natural to work with equations (6), respectively (33) and (35) in order to obtain potentials compatible with the given family. These equations can be used also when the search of the potentials is restricted to a class of theoretical or practical interest.

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