STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 3, September 2004

AN INTERPOLATION BASED COLLOCATION METHOD FOR SOLVING THE DIRICHLET PROBLEM

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Abstract. In this paper we study the numerical solution of a boundary integral equation reformulation of the Dirichlet problem. We give a brief outline of both this problem and its solvability and of a collocation method based on interpolation. We conclude the paper by giving an error analysis of this collocation method.

1. **The Exterior Dirichlet Problem**

We will study only the *exterior* Dirichlet problem, but would like to mention that all the results hold for the *interior* Dirichlet problem, as well, since their integral equation reformulations are very similar.

Let D denote a bounded open simply-connected region in $\mathbb{R}^3$, and let S denote its boundary. Let $\overline{D} = D \cup S$ and denote by $D_e = \mathbb{R}^3 - \overline{D}$ the region complementary to D. Let $\overline{D}_e = D_e \cup S$. At a point $P \in S$, let $\mathbf{n}_P$ denote the unit normal directed into D, provided that such a normal exists. Also assume that S is a piecewise smooth surface that can be decomposed into a finite union of smooth surfaces intersecting each other along common edges at most. In addition, assume that S has a triangulation $\mathcal{T}_n = \{\Delta_{n,k} \mid 1 \leq k \leq n\}$ with mesh size h (such a triangulation can be obtained as the image of a composition of bijections m_k from the unit simplex σ onto a planar triangle Δ_k and bijections F_j from a right triangle onto each smooth piece S_j of S; for details, see Micula [6, Chapter 2]).

Received by the editors: 08.11.2004.

The Exterior Dirichlet Problem. Find $u \in C^1(\overline{D}_e) \cap C^2(D_e)$ that satisfies

$$\begin{aligned} \Delta u(P) &= 0,\ P \in D_e \\ u(P) &= f(P),\ P \in S \\ u(P) = O(P^{-1}),\ \frac{\partial u(P)}{\partial r} = O(|P|^{-2}) &\quad \text{, as} \quad r = |P| \to \infty \text{ uniformly in } \frac{P}{|P|} \end{aligned} \tag{1}$$

with $f \in C(S)$ a given boundary function.

The boundary value problem (1) has been studied extensively (see Mikhlin [8], Günter [5], Colton [4]). Here we only give a very brief outlook at results on the solvability of the problem (1).

The Divergence Theorem (see Atkinson [2, Theorem 7.1.2]) can be used to obtain a representation formula for harmonic functions.

We seek a solution of (1) in the form of a *double layer potential*

$$u(A) = \int_S \rho(Q) \cdot \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|A-Q|} \right] dS_Q, \quad A \in D_e \tag{2}$$

Using a limiting argument, we obtain the second kind integral equation

$$2\pi\rho(P) - \int_S \rho(Q) \cdot \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] dS_Q = f(P), \quad P \in S \tag{3}$$

The kernel function in (3) is given by

$$\frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] = \frac{\mathbf{n}_Q \cdot (P-Q)}{|P-Q|^3} = \frac{\cos\theta_Q}{|P-Q|^2} \tag{4}$$

where θ_Q denotes the angle between $\mathbf{n}_Q$ and $(P-Q)$. Equation (3) can now be written as

$$\rho(P) - \frac{1}{2\pi} \int_S \rho(Q) \cdot \frac{\cos\theta_Q}{|P-Q|^2}\, dS_Q = \hat{f}(P), \quad P \in S \tag{5}$$

where $\hat{f}(P) = \dfrac{1}{2\pi} f(P)$. For simplicity, we will write $f(P)$ instead of $\hat{f}(P)$.

Write the equation (5) in operator form:

$$(\mathcal{I} - K)\rho = f \tag{6}$$

We have (see Mikhlin [8, Chapters 12 and 16]):

Theorem 1.1. Let S be a C^2 surface. Then the equation (6) has a unique solution $\rho \in X$ for each given function $f \in X$, with $X = C(S)$ or $X = L^2(S)$.

Theorem 1.2. Let S be a smooth surface with $\overline{D_e}$ a region to which the Divergence Theorem can be applied. Assume the function $f \in C(S)$. Then, the Dirichlet problem (1) has a unique solution $u \in C^\infty(D_e)$.

2. A Collocation Method

We will use a collocation method where the collocation nodes are the interpolation (of order r) nodes, chosen the following way:

$$q_{i,j} = \left(\frac{i + (r-3i)\alpha}{r}, \frac{j + (r-3j)\alpha}{r} \right), \quad i, j \geq 0, \quad i + j \leq r \tag{7}$$

for some $0 < \alpha < 1/3$ (these are points interior to the unit simplex, but they get mapped into points interior to each triangle in $\mathcal{T}_n$). For corresponding *Lagrange* functions (see Micula [6, pg. 7-11]), for $g \in C(S)$ define an operator $\mathcal{P}_n$ by

$$\mathcal{P}_n g(P) = \sum_{j=1}^{f_r} g\left(m_k(q_j)\right) l_j(s,t), \quad (s,t) \in \sigma, \quad P = m_k(s,t) \in \Delta_k \tag{8}$$

This interpolates $g(P)$ over each triangular element $\Delta_k \in S$, with the interpolating function polynomial in the parametrization variables s and t.

Define a collocation method with (7). Denote $v_{k,j} = m_k(q_i)$. Substitute

$$\begin{aligned} \rho_n(P) &= \sum_{j=1}^{f_r} \rho_n\left(v_{k,j}\right) l_j(s,t) \\ P &= m_k(s,t) \in \Delta_k, \quad k = 1, ..., n \end{aligned} \tag{9}$$

into (5). To determine the values $\{\rho_n(v_{k,j})\}$, force the equation resulting from the substitution to be true at the collocation nodes $\{v_1, ..., v_{nf_r}\}$. This leads to the linear system

$$\begin{aligned} \rho_n(v_i) \quad - \quad & \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{j=1}^{f_r} \rho_n(v_{k,j}) \int_\sigma \frac{\cos\theta_{v_{k,j}}}{\left|v_i - m_k(s,t)\right|^2} \\ \cdot \quad & \left|(D_s m_k \times D_t m_k)(s,t)\right| d\sigma = f(v_i), \quad i = 1, ..., nf_r \end{aligned} \tag{10}$$

which we write abstractly as

$$(\mathcal{I} - P_n\mathcal{K})\rho_n = \mathcal{P}_n f \tag{11}$$

which will be compared to (6). We have the following result.

Theorem 2.1. Let S be a C^2 surface as described earlier, with $F_j \in C^{r+2}$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I} - P_n\mathcal{K}$ are invertible on $L^\infty(S)$ and have uniformly bounded inverses. For the solution ρ of (6) and the solution ρ_n of (10)

$$\|\rho - \rho_n\|_\infty \leq \left\|(\mathcal{I} - P_n\mathcal{K})^{-1}\right\| \cdot \|\rho - \mathcal{P}_n\rho\|_\infty , \quad n \geq n_0 \tag{12}$$

Furthermore, if $f \in C^{r+1}(S)$, then

$$\|\rho - \rho_n\|_\infty = O(h^{r+1}), \quad n \geq n_0 \tag{13}$$

For the proof, see, for example, Atkinson [1].

So interpolation of order r, leads to an error of order $O(h^{r+1})$. But superconvergent methods can be developed. Next, we want to explore in more detail the collocation method based on piecewise constant interpolation (the centroid method) and show that it is superconvergent at the collocation points. Define the operator $\mathcal{P}_n$ by

$$\mathcal{P}_n g(P) = g(P_k),\, P \in \Delta_k,\, k = 1, ..., n \tag{14}$$

for $g \in C(S)$. Then, $\mathcal{P}_n$ is a bounded operator on $C(S)$ with $\|\mathcal{P}_n\| = 1$. Define a collocation method with (14). Substitute

$$\rho_n(P) = \rho_n(P_k),\, P = m_k(s,t) \in \Delta_k,\, k = 1, ..., n \tag{15}$$

into (5). To determine the values $\{\rho_n(P_k)\}$, force the equation resulting from the substitution to be true at the collocation nodes $\{P_k \mid k = 1, ..., n\}$. This leads to the linear system

$$\begin{aligned} \rho_n(P_i) \quad + \quad & \frac{1}{2\pi}\sum_{k=1}^{n}\rho_n(P_k)\cdot\int_\sigma \frac{\cos\theta_{Q_k}}{|P_k - m_k(s,t)|^2} \\ \cdot \quad & |(D_s m_k \times D_t m_k)(s,t)| \; d\sigma = f(P_k),\, i = 1, ..., n \end{aligned} \tag{16}$$

which can be rewritten abstractly as

$$(\mathcal{I} + P_n\mathcal{K})\,\rho_n = \mathcal{P}_n f \tag{17}$$

which will be compared to (6).

By Theorem 2.1., for the true solution ρ of (6) and the solution ρ_n of the collocation equation (17), we have

$$\|\rho - \rho_n\|_\infty = O(h), \quad n \geq n_0 \tag{18}$$

For $g \in C(\sigma)$, consider the interpolation formula (14), which has degree of precision 0. Integrating it over σ, we obtain

$$\int_\sigma g(s,t)\ d\sigma \approx \int_\sigma \mathcal{L}_\tau g(s,t)\ d\sigma = \frac{1}{2} g\left(\frac{1}{3}, \frac{1}{3}\right) \tag{19}$$

which has degree of precision 1.

For $\tau \subset \mathbb{R}^2$, a planar triangle and for a function $g \in C(\tau)$, the function

$$\mathcal{L}_\tau g(x,y) = g\left(m_\tau\left(\frac{1}{3}, \frac{1}{3}\right)\right) = g(P_\tau) \tag{20}$$

is the constant polynomial interpolating g at the node $m_\tau\left(\frac{1}{3}, \frac{1}{3}\right) = P_\tau$ (the centroid of τ). We have the following.

Lemma 2.2. Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Let $g \in C^2(\tau)$. Let $\Phi \in L^1(\tau)$ be differentiable with the first derivatives $D_x\Phi$, $D_y\Phi \in L^1(\tau)$. Then

$$\left|\int_\tau \Phi(x,y)\,(\mathcal{I} - L_\tau)\,g(x,y)\ d\tau\right| \leq ch^2\left[\int_\tau (|\Phi| + |D\Phi|)\ d\tau\right] \cdot \max_\tau \left\{|Dg|, |D^2|g\right\} \tag{21}$$

For the proof, see Micula [6, pg 74-75].

This result can be extended to general triangles, provided

$$\sup_n \left[\max_{\Delta_{n,k}\in\mathcal{T}_n} r(\Delta_{n,k})\right] < \infty \tag{22}$$

where

$$r(\tau) = \frac{h(\tau)}{h^*(\tau)} \tag{23}$$

with $h(\tau)$ and $h^*(\tau)$ denoting the diameter of τ and the radius of the circle inscribed in τ, respectively.

Corollary 2.3. Let τ be a planar triangle of diameter h, let $g \in C^2(\tau)$, and let $\Phi \in L^1(\tau)$ with both first derivatives in $L^1(\tau)$. Then

$$\begin{aligned} \left| \int_\tau \Phi(x,y)(\mathcal{I} - L_\tau)g(x,y) \right| &\leq c\,(r(\tau))\,h^2 \left[\int_\tau (|\Phi| + |D\Phi|)\ d\tau \right] \\ &\cdot \max_\tau \left\{ \|Dg\|_\infty, \|D^2 g\|_\infty \right\} \end{aligned} \tag{24}$$

where $c\,(r(\tau))$ is some multiple of $r(\tau)$ of (23).

Since formula (22) has degree of precision 1 (odd) over σ, extending it to a square would not improve the degree of precision, which means the same error bound as in Lemma 2.2 is true for a parallelogram formed by two symmetric triangles.

We want to apply the above results to the individual subintegrals in

$$\begin{aligned} \mathcal{K}g(P_i) &= \frac{1}{2\pi} \sum_{k=1}^{n} \int_\sigma \frac{\cos\theta_{Q_k}}{|P_k - m_k(s,t)|^2} \rho\,(m_k(s,t)) \\ &\cdot |(D_s m_k \times D_t m_k)\,(s,t)|\ d\sigma \end{aligned} \tag{25}$$

with the role of g played by $\rho\,(m_k(s,t))\,|(D_s m_k \times D_t m_k)\,(s,t)|$, and the role of Φ played by $\dfrac{\cos\theta_{Q_k}}{|P_k - m_k(s,t)|^2}$. For the derivatives of this last function, we have

Theorem 2.4. Let i be an integer and S be a smooth C^{i+1} surface. Then

$$\left| D_Q^i \left(\frac{\cos\theta_Q}{|P-Q|^2} \right) \right| \leq \frac{c}{|P-Q|^{i+1}}, \quad P \neq Q \tag{26}$$

with c a generic constant independent of P and Q.

For details of the proof, see Micula [6, pg.76].

For the error at the collocation node points, we have the following.

Theorem 2.5. Assume the hypotheses of Theorem 2.1, with each $F_j \in C^2$. Assume $\rho \in C^2$. Assume the triangulation $\mathcal{T}_n$ of S satisfies (22) and is symmetric. For those integrals in (25) for which $P_i \in \Delta_k$, assume that all such integrals are evaluated with an error of $O(h^2)$. Then

$$\max_{1\leq i\leq n} |\rho(P_i) - \widehat{\rho}_n(P_i)| \leq ch^2 \log h \tag{27}$$

Proof. We will bound

$$\max_{1\leq i\leq n} |\mathcal{K}(I-P)_n u(v_i)|$$

For a given node point v_i, denote Δ^* the triangle containing it and denote:

$$\mathcal{T}_n^* = \mathcal{T}_n - \{\Delta^*\}$$

By our assumption, the error in evaluating the integral of (25) over Δ^* will be $O(h^2)$.

Partition $\mathcal{T}_n^*$ into parallelograms to the maximum extent possible. Denote by $\mathcal{T}_n^{(1)}$ the set of all triangles making up such parallelograms and let $\mathcal{T}_n^{(2)}$ contain the remaining triangles. Then

$$\mathcal{T}_n^* = \mathcal{T}_n^{(1)} \cup \mathcal{T}_n^{(2)}.$$

It is easy to show that the number of triangles in $\mathcal{T}_n^{(1)}$ is $O(n) = O(h^{-2})$, and the number of triangles in $\mathcal{T}_n^{(2)}$ is $O(\sqrt{n}) = O(h^{-1})$.

It can be shown that all but a finite number of the triangles in $\mathcal{T}_n^{(2)}$, bounded independent of n, will be at a minimum distance from v_i. That means that the triangles in $\mathcal{T}_n^{(2)}$ are "far enough" from v_i, so that the function $G(v_i, Q)$ is uniformly bounded for Q being in a triangle in $\mathcal{T}_n^{(2)}$ (where we denote by $G(P,Q) = \dfrac{\cos\theta_Q}{|P-Q|^2}$).

First, consider the contribution to the error coming from the triangles in $\mathcal{T}_n^{(2)}$. By Lemma 2.2. the error over each such triangle is $O\left(h^2\|D^2 g\|_\infty\right)$, since the area of each triangle is $O(h^2)$ and using our earlier observation. Having $O(h^{-1})$ such triangles in $\mathcal{T}_n^{(2)}$, the total error coming from triangles in $\mathcal{T}_n^{(2)}$ is $O\left(h^3\|D^2 g\|_\infty\right)$.

Next, consider the contribution to the error coming from triangles in $\mathcal{T}_n^{(1)}$. By Lemma 2.2., the error will be of size $O(h^2)$ multiplied times the integral over each such parallelogram of the maximum of the first derivatives of $G(v_i, Q)$ with respect to Q. Combining these we will have a bound

$$ch^2 \int\limits_{S-\Delta^*} (|G| + |DG|)\, dS_Q \tag{28}$$

By Theorem 2.4., the quantity in (28) is bounded by

$$ch^2 \int_{S-\Delta^*} \left(\frac{1}{|P-Q|} + \frac{1}{|P-Q|^2} \right) dS_Q \tag{29}$$

Using a local representation of the surface and then using polar coordinates, the expression in (29) is of order

$$ch^2 \left(h + \log h \right)$$

Thus, the error arising from the triangles in $\mathcal{T}_n^{(1)}$ is $O(h^2 \log h)$. Combining the error arising from the integrals over Δ^*, $\mathcal{T}_n^{(1)}$, and $\mathcal{T}_n^{(2)}$, we have (27).

□

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