STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 3, September 2004

AN INTEGRAL EQUATION WITH MODIFIED ARGUMENT

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Abstract. By the fixed point theorem given in the first part of Rus [3] and an idea of Sotomayor [9], a theorem of differentiability of the solution of the equation

$$x(t) = \int_a^b K(t, s, x(s), x(\varphi(s)))ds + g(t), \quad t \in [\alpha, \beta]$$

is given.

1. **Notations and preliminaries**

Let X be a nonempty set, $A : X \to X$ an operator and we shall use the following notation:

$$F_A := \{x \in X |\ A(x) = x\} \text{ - the fixed point set of } A.$$

Definition 1.1. (Rus [6] or [7]) Let (X, d) be a metric space. An operator $A : X \to X$ is *Picard operator* if there exists $x^* \in X$ such that:

(a) $F_A = \{x^*\}$

(b) the sequence $(A^n(x_0))_{n\in N}$ converges to x^*, for all $x_0 \in X$.

Definition 1.2. (Rus [6] or [7]) Let (X, d) be a metric space. An operator $A : X \to X$ is *weakly Picard operator* if the sequence $(A^n(x_0))_{n\in N}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

If A is a weakly Picard operator, then we consider the following operator

$$A^\infty : X \to X, \quad A^\infty(x) = \lim_{n\to\infty} A^n(x)$$

Received by the editors: 01.06.2004.

2000 *Mathematics Subject Classification.* 45G10, 47H10.

Key words and phrases. Fredholm integral equation with modified argument, Picard operator, data dependence.

It is clear that $A^{\infty}(X) = F_A$.

In the section 2 we need the following results (see [4] and [3]).

Perov's theorem. *Let* (X, d), *with* $d(x, y) \in R^m$, *be a complete generalized metric space and* $A : X \to X$ *an operator. We suppose that there exists a matrix* $Q \in M_{mm}(R_+)$, *such that*

(i) $d(A(x), A(y)) \leq Qd(x, y)$, *for all* $x, y \in X$;

(ii) $Q \to 0$ *as* $n \to \infty$.

Then

(a) $F_A = \{x^*\}$,

(b) $A^n(x) \to x^*$ *as* $n \to \infty$ *and*

$$d(A^n(x), x^*) \leq (I - Q)^{-1} Q^n d(x_0, A(x_0)).$$

Rus theorem. (Rus [3]) *Let* (X, d) *be a metric space (generalized or not) and* (Y, ρ) *be a complete generalized metric space* ($\rho(x, y) \in R^m$).

Let $A : X \times Y \to X \times Y$ *be a continuous operator. We suppose that:*

(i) $A(x, y) = (B(x), C(x, y))$, *for all* $x \in X$, $y \in Y$;

(ii) $B : X \to X$ *is a weakly Picard operator;*

(iii) There exists a matrix $Q \in M_{mm}(R_+)$, $Q^n \to 0$ *as* $n \to \infty$, *such that*

$$\rho(C(x, y_1), C(x, y_2)) \leq Q\rho(y_1, y_2),$$

for all $x \in X$, y_1 *and* $y_2 \in Y$.

Then the operator A is weakly Picard operator. Moreover, if B is Picard operator, then A is Picard operator.

In the section 3 we need the following definition and result (see [8]).

Definition 1.3. (Rus [8]) A matrix $Q \in M_{nn}(\mathbb{R})$ converges to zero if Q^k converges to the zero matrix as $k \to \infty$.

Theorem 1.1. (Rus [8]) *Let* $Q \in M_{nn}(\mathbb{R}_+)$. *The following statements are equivalent:*

(i) $Q^k \to 0$ *as* $k \to \infty$;

(ii) The eigenvalues λ_k, $k = \overline{1,n}$ *of the matrix* Q, *verify the condition* $|\lambda_k| < 1$, $k = \overline{1,n}$;

(iii) The matrix $I - Q$ *is non-singular and* $(I-Q)^{-1} = I + Q + \cdots + Q^n + \dots$.

2. The main result

We consider the following Fredholm integral equation with modified argument

$$x(t) = \int_a^b K(t, s, x(s), x(\varphi(s)))ds + g(t), \quad t \in [\alpha, \beta], \tag{1}$$

where $\alpha, \beta \in R$, $\alpha \leq \beta$, $a, b \in [\alpha, \beta]$, $g \in C([\alpha, \beta], R^m)$, $K \in C([\alpha, \beta] \times [\alpha, \beta] \times R^m \times R^m, R^m)$, $x \in C([\alpha, \beta], R^m)$ and $\varphi \in C([\alpha, \beta], [\alpha, \beta])$.

We have

Theorem 2.1. *We suppose that there exists* $Q \in M_{mm}(R_+)$ *such that:*

(i) $[(\beta - \alpha)Q]^n \to 0$ *as* $n \to \infty$;

(ii)
$$\begin{pmatrix} |K_1(t,s,u,v) - K_1(t,s,w,z)| \\ \dots \\ |K_m(t,s,u,v) - K_m(t,s,w,z)| \end{pmatrix} \leq Q \begin{pmatrix} |u_1 - w_1| + |v_1 - z_1| \\ \dots \\ |u_m - w_m| + |v_m - z_m| \end{pmatrix}$$
for all $u, v, w, z \in R^m$, $t, s \in [\alpha, \beta]$.

Then

(a) the equation (1) has in $C([\alpha, \beta], R^m)$ *a unique solution,* $x^*(\cdot, a, b)$;

(b) for all $x^0 \in C([\alpha, \beta], R^m)$ *the sequence* $(x^n)_{n \in N}$, *defined by*

$$x^{n+1}(t; a, b) := \int_a^b K(t, s, x^n(s; a, b), x^n(\varphi(s); a, b))ds + g(t)$$

converges uniformly to x^*, *for all* $t, a, b \in [\alpha, \beta]$, *and*

$$\begin{pmatrix} |x_1^n(t;a,b) - x_1^*(t;a,b)| \\ \dots \\ |x_m^n(t;a,b) - x_m^*(t;a,b)| \end{pmatrix} \leq$$

$$\leq [I - (\beta - \alpha)Q]^{-1}[(\beta - \alpha)Q]^n \begin{pmatrix} |x_1^0(t;a,b) - x_1^1(t;a,b)| \\ \dots \\ |x_m^0(t;a,b) - x_m^1(t;a,b)| \end{pmatrix}$$

(c) the function

$$x^* : [\alpha,\beta]\times[\alpha,\beta]\times[\alpha,\beta]\to R^m, \quad (t,a,b)\to x^*(t;a,b)$$

is continuous;

(d) if $K(t,s,\cdot,\cdot)\in C^1(R^m\times R^m,R^m)$, for all $t,s\in[\alpha,\beta]$, then $x^*(t;\cdot,\cdot)\in C^1([\alpha,\beta]\times[\alpha,\beta],R^m)$, for all $t\in[\alpha,\beta]$.

Proof. Let $\|\cdot\|$ be a generalized Chebyshev norm on $X := C([\alpha,\beta]^3,R^m)$ i.e.

$$\|x\| := \begin{pmatrix} \|x_1\|_\infty \\ \dots \\ \|x_m\|_\infty \end{pmatrix}.$$

Let we consider the operator $B : X\to X$ defined by

$$B(x)(t;a,b) := \int_a^b K(t,s,x(s;a,b),x(\varphi(s);a,b))ds$$

for all $t,a,b\in[\alpha,\beta]$.

From (i) and (ii) and the Perov's theorem we have (a)+(b)+(c).

(d) Let we prove that there exists $\dfrac{\partial x^*}{\partial a}$ and $\dfrac{\partial x^*}{\partial a}\in X$.

If we suppose that there exists $\dfrac{\partial x^*}{\partial a}$, then from (1) we have

$$\begin{aligned}\frac{\partial x^*(t;a,b)}{\partial a} &= -K(t,a,x^*(a;a,b),x^*(\varphi(a);a,b))+ \\ &+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(\varphi(s);a,b))}{\partial x_i}\right)\frac{\partial x^*(s;a,b)}{\partial a}+\right. \\ &\left.+\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(\varphi(s);a,b))}{\partial x_i}\right)\frac{\partial x^*(\varphi(s);a,b)}{\partial a}\right]ds.\end{aligned}$$

This relation suggest to consider the following operator

$$C : X\times X\to X,$$

$$C(x,y)(t;a,b) := -K(t,a,x(a;a,b),x(\varphi(a);a,b))+ \quad (2)$$

$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x(s;a,b),x(\varphi(s);a,b))}{\partial x_i}\right) y(s;a,b)+\right.$$

$$\left.+\left(\frac{\partial K_j(t,s,x(s;a,b),x(\varphi(s);a,b))}{\partial x_i}\right) y(\varphi(s);a,b)\right] ds.$$

From (ii), we remark that

$$\left(\left|\frac{\partial K_j(t,s,u,v)}{\partial x_i}\right|\right) \leq Q \quad (3)$$

for all $t,s \in [\alpha,\beta]$ and $u,v \in R^m$.

From (2) and (3) it follows that

$$\|C(x,y_1)-C(x,y_2)\| \leq (\beta-\alpha)Q,$$

for all $x,y_1,y_2 \in X$.

If we take the operator

$$A : X\times X \to X\times X, \quad A=(B,C),$$

then we are in the conditions of the Rus theorem. From this theorem, the operator A is a Picard operator and the sequences

$$x^{n+1}(t;a,b) = \int_a^b K(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))ds + g(t)$$

$$y^{n+1}(t;a,b) := -K(t,a,x^n(a;a,b),x^n(\varphi(a);a,b))+$$

$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))}{\partial x_i}\right) y^n(s;a,b)+\right.$$

$$\left.+\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))}{\partial x_i}\right) y^n(\varphi(s);a,b)\right] ds$$

converges uniformly (with respect to $t,a,b \in [\alpha,\beta]$) to $(x^*,y^*) \in F_A$, for all $x^0,y^0 \in X$.

If we take $x^0=y^0=0$, then $y^1 = \dfrac{\partial x^1}{\partial a}$. By induction we prove that $y^n = \dfrac{\partial x^n}{\partial a}$. Thus

$$x^n \overset{unif.}{\longrightarrow} x^* \text{ as } n\to\infty,$$

$$\frac{\partial x^n}{\partial a} \xrightarrow{unif.} y^* \text{ as } n \to \infty.$$

These imply that there exists $\dfrac{\partial x^*}{\partial a}$ and $\dfrac{\partial x^*}{\partial a} = y^*$.

By a similar way we prove that there exists $\dfrac{\partial x^*}{\partial b}$. □

3. **Example**

In what follows we consider the following system of Fredholm integral equations

$$\begin{cases} x_1(t) = \displaystyle\int_a^b \left[\frac{1}{8}(t+s)x_1(s) + \frac{1}{4}x_1(s/2)\right] ds + 1 - \cos t \\ x_2(t) = \displaystyle\int_a^b \left[\frac{1}{2}x_1(x) + \frac{2t+s}{4}x_2(s) + \frac{3}{4}x_2(s/2)\right] ds + \sin t \end{cases}, \tag{4}$$

$t, a, b \in [0,1]$, where $a, b \in [0,1]$, $g \in C([0,1], \mathbb{R}^2)$, $g(t) = (g_1(t), g_2(t))$, $g_1(t) = 1 - \cos t$, $g_2(t) = \sin t$, $K \in C([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$,

$$K(t, s, x(s), x(\varphi(s))) = (K_1(t, s, x(s), x(\varphi(s))), K_2(t, s, x(s), x(\varphi(s)))),$$

$$K_1 = \frac{1}{8}(t+s)x_1(s) + \frac{1}{4}x_1(s/2), \quad K_2 = \frac{1}{2}x_1(x) + \frac{2t+s}{4}x_2(s) + \frac{3}{4}x_2(s/2),$$

$\varphi \in C([0,1], [0,1])$, $\varphi(s) = s/2$ and $x \in C([0,1], \mathbb{R}^2)$.

From the condition (ii) of the theorem 2.1 we have

$$\begin{pmatrix} |K_1(t,s,x(s),x(s/2)) - K_1(t,s,x(s),z(s/2))| \\ |K_2(t,s,x(s),x(s/2)) - K_2(t,s,x(s),z(s/2))| \end{pmatrix} \le$$

$$\le \begin{pmatrix} 1/4 & 0 \\ 1/2 & 3/4 \end{pmatrix} \begin{pmatrix} |x_1(s) - z_1(s)| + |x_1(s/2) - z_1(s/2)| \\ |x_2(s) - z_2(s)| + |x_2(s/2) - z_2(s/2)| \end{pmatrix}, \quad t, s \in [0,1],$$

which lead to matrix

$$Q = \begin{pmatrix} 1/4 & 0 \\ 1/2 & 3/4 \end{pmatrix}, \quad Q \in M_{22}(\mathbb{R}_+),$$

that according to the theorem 1.1 and definition 1.3, converges to zero,

Therefore the conditions of the theorem 2.1 are satisfies and we have

- the system of equations (4) has in $C([0,1], \mathbb{R}^2)$ a unique solution $x^*(\cdot, a, b)$;

- for all $x^0 \in C([0,1],\mathbb{R}^2)$ the sequence $(x^n)_{n\in\mathbb{N}}$, defined by

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(\varphi(s);a,b))ds + g(t)$$

converges uniformly to x^*, for all $t,a,b \in [0,1]$, and

$$\begin{pmatrix} |x_1^n(t;a,b) - x_1^*(t;a,b)| \\ \dots \\ |x_m^n(t;a,b) - x_m^*(t;a,b)| \end{pmatrix} \le [I-Q]^{-1}Q^n \begin{pmatrix} |x_1^0(t;a,b) - x_1^1(t;a,b) \\ \dots \\ |x_m^0(t;a,b) - x_m^1(t;a,b) \end{pmatrix}$$

- the function

$$x^* : [0,1]\times[0,1]\times[0,1] \to \mathbb{R}^2, \quad (t;a,b) \to x^*(t;a,b)$$

is continuous;

- if $K(t,s,\cdot,\cdot) \in C^1(\mathbb{R}^2\times\mathbb{R}^2,\mathbb{R}^2)$, for all $t,s \in [0,1]$, then $x^*(t;\cdot,\cdot) \in C^1([0,1]\times[0,1],\mathbb{R}^2)$, for all $t \in [0,1]$.

References

[1] M. Ambro, *Aproximarea soluţiilor unei ecuaţii integrale cu argument modificat*, Studia Univ. Babeş-Bolyai, Math., **2**(1978), 26-32.

[2] D. Guo, V. Lakshmikantham, X. Liu, *Nonlinear integral equations in abstract spaces*, Kluwer, London, 1996.

[3] I. A. Rus, *A Fiber generalized contraction theorem and applications*, Academie Roumaine Filiale de Cluj-Napoca, Mathematica, Tome **41**(64), Nr. 1, 1999, 85-90.

[4] I. A. Rus, *Generalized contractions*, Univ. of Cluj-Napoca, Preprint Nr. 3, 1983, 1-130.

[5] I. A. Rus, C. Iancu, *Modelare matematică*, Transilvania Press, Cluj-Napoca, 2000.

[6] I. A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Caroline, **34**, 3(1993), 769-773.

[7] I. A. Rus, *Picard operators and applications*, Babeş-Bolyai Univ. of Cluj-Napoca, Preprint Nr. 3, 1996.

[8] I. A. Rus, *Principii şi aplicaţii ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979.

[9] J. Sotomayor, *Smooth dependence of solution of differential equation on initial data: a simple proof*, Bol. Soc. Brasil, **4**, 1(1973), 55-59.

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