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ON THE CONVERGENCE OF COLLOCATION SPLINE METHODS FOR INTEGRAL DELAY PROBLEMS

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Abstract. In some recent works we proposed a collocation method by deficient splines to approximate the solution of Neutral Delay Differential Equations and Volterra Delay Integral Equations. In this work we extend that method to integro-differential equations. The existence and uniqueness of the numerical solution is proved. Consistency and convergence of this method are studied.

1. **The problem**

In this work we present some remarks about the convergence of a collocation spline method for a problem which is the synthesis of models recently studied in collaboration with Professor Georghe Micula.

Precisely we consider the following non linear first-order Fredholm integro-differential problem with delay:

$$\begin{gathered} y'(x) = f(x, y(x), y(g(x)), \textstyle\int_0^T K(x, t, y(t), y(g(t)))dt), \quad x \in [0, T] \\ y(0) = y_0, \quad y(x) = \psi(x), \quad x \in [\alpha, 0], \quad \alpha \le 0, \quad \alpha = \underset{x\in[0,T]}{Inf}(g(x)) \\ \alpha \le g(x) \le x \ , \qquad x \in [\alpha, T] \end{gathered} \tag{1}$$

where $f : [0, T] \times \mathcal{R}^3 \to \mathcal{R}$, $K : [0, T] \times [0, T] \times \mathcal{R}^2 \to \mathcal{R}$, $g \in \mathcal{C}[\alpha, T]$, $\psi \in \mathcal{C}^{m-1}[\alpha, 0]$, $m > 1$, $m \in N$.

(1) can be considered Volterra delay integro-differential problem by replacing the upper limit of integration T by x.

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As usual we write problem (1) in the following form

$$\begin{array}{c} y'(x) = f(x, y(x), y(g(x)), z(x)), \quad x \in [0, T] \\ z(x) = \int_0^T K(x, t, y(t), y(g(t)))dt \\ y(0) = y_0, \quad y(x) = \psi(x), \quad x \in [\alpha, 0], \quad \alpha \leq 0, \quad \alpha = \underset{x \in [0,T]}{Inf}(g(x)) \\ \alpha \leq g(x) \leq x \ , \qquad x \in [\alpha, T] \end{array} \tag{2}$$

In the following we assume $g(x) = x - \tau$, where $\tau \in \mathcal{R}$, $\tau > 0$ is the constant delay. Let $y_\tau = y(x - \tau)$ and $T = M\tau$ for some $M \in \mathcal{N}$.

1. Suppose that $f(x, y, y_\tau, z)$ is a smooth function satisfying the following Lipschitz condition

$||f(x, y_1, y_{\tau 1}, z_1) - f(x, y_2, y_{\tau 2}, z_2)|| \leq$
$L_1(||y_1 - y_2|| + ||y_{\tau 1} - y_{\tau 2}|| + ||z_1 - z_2||)$
$\forall \ (x, y_1, y_{\tau 1}, z_1), \ (x, y_2, y_{\tau 2}, z_2) \in [0, T] \times \mathcal{R}^3$.

2. Suppose also that the kernel $K(x, t, y, y_\tau)$ is a smooth bounded function satisfying the following Lipschitz condition

$||K(x, t, y_1, y_{\tau 1}) - K(x, t, y_2, y_{\tau 2})|| \leq$
$L_2(||y_1 - y_2|| + ||y_{\tau 1} - y_{\tau 2}||)$
$\forall \ (x, t, y_1, y_{\tau 1}), \ (x, t, y_2, y_{\tau 2}) \in [0, T] \times [0, T] \times \mathcal{R}^2$.

In these conditions, the problem (2) has a unique solution (see for example [2]).

To face this mathematical model we propose a numerical model based on direct collocation spline method using the well known advantages of a collocation method and of a spline approximation. In particular we construct splines pertaining to low regularity class and with weak regularity conditions in the junction points.

The collocation allows to recursively define a piecewise approximating polynomial and is characterized (differently from what is suggested by the literature) by the fact that knowledge gathered in previous steps is completely utilized, thus refining the approximating solution, even at price of a heavier computational load.

Let $r \in \mathcal{N}$, $\quad N = rM \quad$ and Δ be the following uniform partition of the interval $[0, T]$:

$\Delta : 0 = x_0 < x_1 < ... < x_k < x_{k+1} < ... < x_N = T \ , \ x_k = kh, \ \ h = \frac{T}{N}.$

Here we approximate the solution of (2) by means of functions pertaining to the class of spline of degree $m \geq 2$ and deficiency 2, denoted by $s : [0, T] \to \mathcal{R}$, $(s \in \mathcal{S}_m,\ s \in \mathcal{C}^{m-2})$.

Precisely, the spline function s is defined in $I_k = [x_k, x_{k+1}]$ as:

$$s_k(x) := \sum_{j=0}^{m-2} s_{k-1}^{(j)}(x_k)(x - x_k)^j/j! + \frac{a_k}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k}{m!}(x - x_k)^m$$

We choose to determine coefficients $a_k,\ b_k$ by the following system of collocation conditions

$$\left\{ \begin{array}{c} s'_k(x_k + \frac{h}{2}) = f(x_k + \frac{h}{2}, s_k(x_k + \frac{h}{2}), s_{k-r}(x_k + \frac{h}{2} - \tau), z_k(x_k + \frac{h}{2})) \\ s'_k(x_{k+1}) = f(x_{k+1}, s_k(x_{k+1}), s_{k-r}(x_{k+1} - \tau), z_k(x_{k+1})) \end{array} \right. \tag{3}$$

where

$$\begin{aligned} z_k(x_k + \frac{h}{2}) \quad &= \quad \sum_{j=0}^{k-1} \int_{jh}^{(j+1)h} K(x_k + \frac{h}{2}, t, s_j(t), s_{j-r}(t - \tau))dt + \\ &\quad + \int_{kh}^{kh + \frac{h}{2}} K(x_k + \frac{h}{2}, t, s_k(t), s_{k-r}(t - \tau))dt \end{aligned}$$

and

$$z_k(x_{k+1}) = \sum_{j=0}^{k} \int_{jh}^{(j+1)h} K(x_k + h, t, s_j(t), s_{j-r}(t - \tau))dt$$

provided that

$$s_k^{(i)}(x_k) = \lim_{x \to x_k} s_{k-1}^{(i)}(x),\ x \in [x_{k-1}, x_k] \text{ for } i = 0, ..., m-2$$

$$s_i(x) = \psi(x), \text{ for } i = -r, ..., -1.$$

Our model is thus reduced to compute the solution of the system (3), through which the spline is determined on the interval I_k.The system can be either non-linear or linear according to $f(x, y(x), y(g(x)), z(x))$.

2. Existence and uniqueness of numerical solution

If we set in $[x_k, x_{k+1}],\ k = 0, 1, ..., N-1$:
$A_k(x) = \sum_{j=0}^{m-2} s_{k-1}^{(j)}(x_k)(x - x_k)^j / j!$,

$$\mathbf{B}_k = \begin{bmatrix} a_k\,(\frac{h}{2})^{m-2} \\ b_k\,(\frac{h}{2})^{m-1} \end{bmatrix}, \qquad \mathbf{Y}_k = \begin{bmatrix} A'_k(x_k + \frac{h}{2}) \\ A'_k(x_{k+1}) \end{bmatrix},$$

$$\mathbf{P} = \frac{1}{(m-2)!}\mathbf{P}_0 \quad \text{with} \quad \mathbf{P}_0 = \begin{bmatrix} 1 & \frac{1}{m-1} \\ 2^{m-2} & \frac{2^{m-1}}{m-1} \end{bmatrix}$$

$$\mathbf{\Phi}_k(\mathbf{B}_k) = \begin{bmatrix} f(x_k + \frac{h}{2}, A_k(x_k + \frac{h}{2}) + \frac{a_k}{(m-1)!}(\frac{h}{2})^{m-1} + \frac{b_k}{m!}(\frac{h}{2})^m, \\ s_{k-r}(x_k + \frac{h}{2} - \tau), z_k(x_k + \frac{h}{2})) \\ f(x_{k+1}, A_k(x_{k+1}) + \frac{a_k}{(m-1)!}h^{m-1} + \frac{b_k}{m!}h^m, \\ s_{k-r}(x_{k+1} - \tau), z_k(x_{k+1})) \end{bmatrix}$$

then (3) becomes:

$$\mathbf{P}\mathbf{B}_k = \mathbf{\Phi}_k(\mathbf{B}_k) - \mathbf{Y}_k$$

Taking into account that $\mathbf{P}_0$ is non singular $\forall m > 1$, system (3) is equivalent to

$$\mathbf{B}_k = (m-2)!\mathbf{P}_0^{-1}(\mathbf{\Phi}_k(\mathbf{B}_k) - \mathbf{Y}_k) \tag{4}$$

Theorem 1. *Let us consider the nonlinear first-order Fredholm integro-differential equation with delay in (2). If functions f and K satisfy the Lipschitz conditions 1. and 2. and if h is small enough, then there exists a unique spline approximation solution $s(x)$ of the problem (2) given by the above construction.*

Proof. The proof of Theorem 1 consists of showing that (4) defines for all sufficiently small h, a contraction mapping. This comes straightforward from the hypotheses and we omit the details of the proof. □

In general, $z_k(x_k+\frac{h}{2}), z_k(x_{k+1})$ in (3) have to be approximated by numerical quadrature: $z_k(x_k+\frac{h}{2}) \simeq Z_k(x_k+\frac{h}{2}),\ z_k(x_{k+1}) \simeq Z_k(x_{k+1})$, where

$$\begin{aligned} Z_k(x_k+\frac{h}{2}) &= \sum_{l=0}^{k-1}\left[\sum_{j=0}^{n(l)} w_j^{(l)}\, K(x_k+\frac{h}{2}, t_j^{(l)}, s_l(t_j^{(l)}), s_{l-r}(t_j^{(l)}-\tau))\right] + \\ &\quad + \sum_{j=0}^{n(k)} w_j^{(k)}\, K(x_k+\frac{h}{2}, t_j^{(k)}, s_k(t_j^{(k)}), s_{k-r}(t_j^{(k)}-\tau)) \end{aligned}$$

with $x_l \le t_j^{(l)} \le x_{l+1}\,(l=0,1,...,k-1) \qquad x_k \le t_j^{(k)} \le x_k+\frac{h}{2}$

$$Z_k(x_{k+1}) = \sum_{l=0}^{k}\left[\sum_{j=0}^{n(l)} w_j^{(l)}\, K(x_{k+1}, t_j^{(l)}, s_l(t_j^{(l)}), s_{l-r}(t_j^{(l)}-\tau))\right]$$

with $x_l \le t_j^{(l)} \le x_{l+1}(l=0,1,...,k)$ and we assume that $\max_{j,l}\left|w_j^{(l)}\right| \le W < \infty$.

System (3) is then reduced to

$$\mathbf{B}_k = (m-2)!\mathbf{P}_0^{-1}(\mathbf{\Psi}_k(\mathbf{B}_k) - \mathbf{Y}_k) \tag{5}$$

with

$$\mathbf{\Psi}_k(\mathbf{B}_k) = \begin{bmatrix} f(x_k+\frac{h}{2}, s_k(x_k+\frac{h}{2}), s_{k-r}(x_k+\frac{h}{2}-\tau), Z_k(x_k+\frac{h}{2})) \\ f(x_{k+1}, s_k(x_{k+1}), s_{k-r}(x_{k+1}-\tau), Z_k(x_{k+1})) \end{bmatrix}$$

Theorem 2. *Under the assumptions stated above and if h is small enough, there exists a unique solution of system (5).*

Proof. As in Theorem 1, the proof consists of showing that (5) defines for all sufficiently small h, a contraction mapping. □

3. Consistency and convergence of the collocation method

Let $y(x) \in \mathcal{C}^{m+1}[0,T]$, $s_k(x)$ be the deficient spline approximating $y(x)$ in $[x_k, x_{k+1}]$, $(k = 0,1,...,N-1)$ and denote with $e_k(x) = s_k(x) - y(x)$ the error function for $x \in [x_k, x_{k+1}]$.

Considering $y(x) = y(x_k) + y'(x_k)(x - x_k) + ... + y^{(m+1)}(\eta_k)\frac{(x-x_k)^{m+1}}{(m+1)!}$ $(x_k < \eta_k < x)$ then

$$\begin{aligned} e_k(x) &= [\sum_{j=0}^{m-2} \frac{s_{k-1}^{(j)}(x_k)}{j!}(x - x_k)^j + \frac{a_k}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k}{m!}(x - x_k)^m] + \\ &\quad - \left[y(x_k) + y'(x_k)(x - x_k) + ... + y^{(m+1)}(\eta_k)\frac{(x - x_k)^{m+1}}{(m+1)!}\right] \\ (x_k &< \eta_k < x) \end{aligned}$$

consequently

$$\begin{aligned} e_k(x) &= e_k(x_k) + \sum_{j=1}^{m-2} \frac{s_{k-1}^{(j)}(x_k) - y^{(j)}(x_k)}{j!}(x - x_k)^j + \\ &\quad + \frac{a_k - y^{(m-1)}(x_k)}{(m-1)!}(x - x_k)^{m-1} + \frac{b_k - y^{(m)}(x_k)}{m!}(x - x_k)^m + \\ &\quad - \frac{y^{(m+1)}(\eta_k)}{(m+1)!}(x - x_k)^{m+1} \\ (x_k &< \eta_k < x) \end{aligned}$$

If we set for $k = 0, 1, ..., N - 1$

$$\begin{aligned} \beta_{k,m-1} &= \frac{a_k - y^{(m-1)}(x_k)}{h^2} \\ \beta_{k,m} &= \frac{b_k - y^{(m)}(x_k)}{h} \\ \gamma_{k,j} &= \frac{s_{k-1}^{(j)}(x_k) - y^{(j)}(x_k)}{h^{m-j+1}}, \ j = 1, ..., m - 2 \\ \varphi_{k,i}(x) &= \frac{(x - x_k)^i}{h^i} \qquad (i = 1, 2, ...) \end{aligned}$$

and

$$T_k(y(x)) = \frac{y^{(m+1)}(\eta_k)}{(m+1)!}, \ x_k < \eta_k < x$$

then the error becomes

$$\begin{gathered} e_k(x) = e_k(x_k) + h^{m+1} \textstyle\sum_{j=1}^{m-2} \frac{\gamma_{k,j}}{j!}\varphi_{k,j}(x) + \\ + h^{m+1}\left[\frac{\beta_{k,m-1}}{(m-1)!}\varphi_{k,m-1}(x) + \frac{\beta_{k,m}}{m!}\varphi_{k,m}(x) - T_k(y(x))\varphi_{k,m+1}(x)\right] \end{gathered} \tag{6}$$

with

$$\begin{array}{c} e_k'(x) = e_k'(x_k) + h^m \sum_{j=2}^{m-2} \frac{\gamma_{k,j}}{(j-1)!}\varphi_{k,j-1}(x) + \\ +h^m \left[\frac{\beta_{k,m-1}}{(m-2)!}\varphi_{k,m-2}(x) + \frac{\beta_{k,m}}{(m-1)!}\varphi_{k,m-1}(x) - T_k'(y(x))\varphi_{k,m}(x)\right] \end{array} \tag{7}$$

where

$$T_k'(y(x)) = \frac{y^{(m+1)}(\mu_k)}{m!},\ x_k < \mu_k < x$$

Lemma 3. *Let the hypotheses 1. and 2. hold for f and K, then there exists a constant c independent of h such that*

$\sum_{j=1}^{m-2} \gamma_{k,j} = O(h)$ *for all* $k = 0, 1, ..., N-1$

Proof. The proof comes straightforward from Lemma 4.3 [1]. □

Lemma 4. *(i) Let $f(x, y, y_\tau, z)$ have continuous derivatives of order one with respect to y, y_τ, z in $[0, T]$*

(ii) Let $K(x, t, y, y_\tau)$ have continuous derivatives of order one with respect to y, y_τ in T

then $|\beta_{k,m-1}| + |\beta_{k,m}| \leq B$ for all $k = 0, 1, ..., N-1$, where B is a real constant.

Proof. Let $k = 0$, then $e_0(0) = e_0'(0) = 0$ as $x_0 = 0$. We observe that $\varphi_{0,\nu}(\frac{h}{2}) = \frac{1}{2^\nu}$ and $\varphi_{0,\nu}(h) = 1$, $\nu = 1, 2, ..., m+1$, taking account of Lemma 3, from (7) we obtain:

$$\left\{\begin{array}{c} \frac{\beta_{0,m-1}}{(m-2)!}\frac{1}{2^{m-2}} + \frac{\beta_{0,m}}{(m-1)!}\frac{1}{2^{m-1}} = T_0'(y(\frac{h}{2}))\frac{1}{2^m} + e_0'(\frac{h}{2}) + O(h) \\ \frac{\beta_{0,m-1}}{(m-2)!} + \frac{\beta_{0,m}}{(m-1)!} = T_0'(y(h)) + e_0'(h) + O(h) \end{array}\right. \tag{8}$$

In order to prove that (8) has a unique limited solution, we follow Theorem 1 in [2], taking account of the delay terms.

We observe that a simple calculation yields for $k = 0, 1, ..., N-1$, using the hypotheses on f and K

$$\begin{aligned} e_k'(x_k + \frac{h}{2}) \quad = \quad & \frac{\partial}{\partial y} f(x_k + \frac{h}{2}, y_k^*, y_{k\tau}^*, z_k^*)\, e_k(x_k + \frac{h}{2}) + \\ & + \frac{\partial}{\partial y_\tau} f(x_k + \frac{h}{2}, y_k^*, y_{k\tau}^*, z_k^*)\, e_k(x_k + \frac{h}{2} - \tau) + \\ & + \frac{\partial}{\partial z} f(x_k + \frac{h}{2}, y_k^*, y_{k\tau}^*, z_k^*)\, \delta_k(x_k + \frac{h}{2}) \end{aligned}$$

where:

y_k^* is between $y(x_k + \frac{h}{2})$ and $s_k(x_k + \frac{h}{2})$,

$y_{k\tau}^*$ between $y(x_k + \frac{h}{2} - \tau)$ and $s_k(x_k + \frac{h}{2} - \tau)$,

z_k^* between $z(x_k + \frac{h}{2})$ and $z_k(x_k + \frac{h}{2})$,

$e_k(x_k + \frac{h}{2} - \tau) = s_k(x_k + \frac{h}{2} - \tau) - y(x_k + \frac{h}{2} - \tau)$ and

$$\begin{aligned}\delta_k(x_k + \frac{h}{2}) &= \int_{x_0}^{x_k+\frac{h}{2}} [\frac{\partial}{\partial y} K(x_k + \frac{h}{2}, t, y^*(t), y_\tau^*(t-\tau))e(t) + \\ &\quad \frac{\partial}{\partial y_\tau} K(x_k + \frac{h}{2}, t, y^*(t), y_\tau^*(t-\tau))e(t-\tau)]dt\end{aligned}$$

$y^*(t)$ being between $y(t)$ and $s(t)$, $y_\tau^*(t-\tau)$ between $y(t-\tau)$ and $s(t-\tau)$.

In the same way we obtain

$$\begin{aligned}e_k'(x_{k+1}) &= \frac{\partial}{\partial y} f(x_{k+1}, \overline{y}^*, \overline{y}_\tau^*, \overline{z}^*)\, e_k(x_{k+1}) + \\ &\quad + \frac{\partial}{\partial y_\tau} f(x_{k+1}, \overline{y}^*, \overline{y}_\tau^*, \overline{z}^*)\, e_k(x_{k+1} - \tau) + \\ &\quad + \frac{\partial}{\partial z} f(x_{k+1}, \overline{y}^*, \overline{y}_\tau^*, \overline{z}^*)\, \delta_k(x_{k+1})\end{aligned}$$

with suitable $\overline{y}_k^*$, $\overline{y}_{k\tau}^*$, $\overline{z}_k^*$, $\overline{y}^*(t)$, $\overline{y}_\tau^*(t-\tau)$ and an obvious definition of $e_k(x_{k+1} - \tau)$ and $\delta_k(x_{k+1})$.

Consequently we obtain

$$\begin{aligned}e_0'(\frac{h}{2}) &= \frac{\partial}{\partial y} f(\frac{h}{2}, y_0^*, y_{0\tau}^*, z_0^*)\, e_0(\frac{h}{2}) + \\ &\quad + \frac{\partial}{\partial y_\tau} f(\frac{h}{2}, y_0^*, y_{0\tau}^*, z_0^*)\, e_0(\frac{h}{2} - \tau) + \\ &\quad + \frac{\partial}{\partial z} f(\frac{h}{2}, y_0^*, y_{0\tau}^*, z_0^*)\, \delta_0(\frac{h}{2})\end{aligned}$$

and

$$\begin{aligned}e_0'(h) &= \frac{\partial}{\partial y} f(h, \overline{y}_0^*, \overline{y}_{0\tau}^*, \overline{z}_0^*)\, e_0(h) + \\ &\quad + \frac{\partial}{\partial y_\tau} f(h, \overline{y}_0^*, \overline{y}_{0\tau}^*, \overline{z}_0^*)\, e_0(h - \tau) + \\ &\quad + \frac{\partial}{\partial z} f(h, \overline{y}_0^*, \overline{y}_{0\tau}^*, \overline{z}_0^*)\, \delta_0(h)\end{aligned}$$

so that, according to Lemma 1 in [2], the unique solution $\overline{\beta}_{0,m-1}, \overline{\beta}_{0,m}$ of the system

$$\begin{cases} \frac{\beta_{0,m-1}}{(m-2)!}\frac{1}{2^{m-2}} + \frac{\beta_{0,m}}{(m-1)!}\frac{1}{2^{m-1}} = \frac{1}{2^m}T_0'(y(\frac{h}{2})) \\ \frac{\beta_{0,m-1}}{(m-2)!} + \frac{\beta_{0,m}}{(m-1)!} = T_0'(y(h)) \end{cases}$$

can be regarded as the solution $\beta_{0,m-1}, \beta_{0,m}$ of the system (8) for $h \to 0$ and

$$\beta_{0,\nu} = \overline{\beta}_{0,\nu} + O(h), \ \nu = m-1, m.$$

Let $k = 1$, we observe from (6) and (7) that $e_1(x_1) = e_0(x_1) = O(h^{m+1})$ and $e_1'(x_1) = e_0'(x_1) = O(h^{m+1})$, we proceed by induction on k in the same way as for $k = 0$. The proof of the Lemma follows immediately. □

Theorem 5. *Under the assumptions stated in Lemma 4, then there exists a constant C independent of h such that the error function $e(x)$ satisfies for all $x \in [0,T]$ the following inequalities*

$$|e(x)| \leq C\,h^{m+1}$$
$$|e'(x)| \leq C\,h^m$$

Proof. We initially prove the Theorem for $x = x_k$. If we set $M_{m+1} = \max\limits_{k,x\in[0,T]} |T_k'(y(x))|$, then $|T_k(y(x))| \leq M_{m+1}$ for all $x \in [0,T]$; from (6) and Lemma 3 the following relation holds:

$$|e_k(x_k)| \leq \left|e_{k-1}(x_{k-1})\right| + h^{m+1}(c + B + M_{m+1})$$

where c is real constant.

Taking into account that $|e_1(x_1)| \leq h^{m+1}(B + M_{m+1})$ then

$|e_2(x_2)| \leq |e_1(x_1)| + h^{m+1}(c + B + M_{m+1}) \leq h^{m+1}(c + 2(B + M_{m+1}))$,

and

$$|e_k(x_k)| \leq N h^{m+1}(c + B + M_{m+1}) \tag{9}$$

It follows that $|e_k(x_k)| \leq C_1 h^{m+1}$.

Taking into account of (6) we obtain for $x \in [x_k, x_{k+1}]$

$$|e_k(x)| \leq |e_k(x_k)| + h^{m+1}\sum_{j=1}^{m-2}\frac{|\gamma_{k,j}|}{j!}\varphi_{k,j}(x) +$$
$$+ h^{m+1}\left|\frac{\beta_{k,m-1}}{(m-1)!}\varphi_{k,m-1}(x) + \frac{\beta_{k,m}}{(m)!}\varphi_{k,m}(x) - T_k(y(x))\varphi_{k,m+1}(x)\right|$$

from (9), Lemma 3 and Lemma 4 and $|\varphi_{k,j}(x)| \leq 1,\ j = 1, ..., m+1$ we obtain

$$|e_k(x)| \leq N h^{m+1}(c + B + M_{m+1}) + c\, h^{m+1} \sum_{j=1}^{m-2} \frac{1}{j!} + $$
$$+ h^{m+1} \left| B(\frac{1}{(m-1)!} + \frac{1}{(m)!}) + M_{m+1} \right|$$

it follows $|e_k(x)| \leq C\, h^{m+1}$.

Analogously we obtain

$$|e'_k(x)| \leq N h^{m}(c + B + M_{m+1}) + c\, h^{m} \sum_{j=2}^{m-2} \frac{1}{(j-1)!} + $$
$$+ h^{m} \left| B(\frac{1}{(m-2)!} 1 + \frac{1}{(m-1)!}) + M_{m+1} \right|$$

It follows $|e'_k(x)| \leq C_1\, h^{m}$.

Because any upper bound for $|e_k(x)|$ and for $|e'_k(x)|$ is independent of k, the thesis follows. □

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