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ON STRONGLY NONLINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS OF DIVERGENCE FORM

LÁSZLÓ SIMON

Abstract. We consider initial boundary value problems for second order strongly nonlinear parabolic equations where also the main part contains functional dependence on the unknown function.

Introduction

This investigation was motivated by works [4], [5] of M. Chipot on "nonlocal evolution problems" for the equation

$$D_t u - \sum_{i,j=1}^{n} D_i[a_{ij}(l(u(\cdot,t))D_i u] + a_0(l(u(\cdot,t))u = f \text{ in } \Omega \times R^+ \qquad (0.1)$$

where $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary,

$$\sum_{i,j=1}^{n} a_{ij}(\zeta)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for all } \xi \in R^n, \quad \zeta \in R$$

with some constant $\lambda > 0$,

$$l(u(\cdot,t)) = \int_\Omega g(x)u(x,t)dx$$

with a given function $g \in L^2(\Omega)$. Existence and asymptotic properties (as $t \to \infty$) of solutions of initial-boundary value problems for (0.1) were proved. That problem was motivated by diffusion process (for heat or population), where the diffusion coefficient depends on a nonlocal quantity.

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Our aim is to consider similar problems for quasilinear parabolic functional differential equations of the form

$$D_t u - \sum_{i=1}^{n} D_i[a_i(t,x,u(t,x),Du(t,x);u)] + a_0(t,x,u(t,x),Du(t,x);u) + \tag{0.2}$$

$$b(t,x,u(t,x);u) = f \text{ in } Q_{T_0} = (0,T_0)\times\Omega$$

with homogeneous Dirichlet boundary and initial conditions, where the functions

$$a_i : Q_{T_0} \times R^{n+1} \times L^p(0,T_0;V) \to R$$

(with $V = W_0^{1,p}(\Omega)$, $2 \le p < \infty$) satisfy conditions which are generalizations of conditions for strongly nonlinear parabolic differential equations, considered in [3], [7], [8] by using the theory of monotone type operators; a_i have polynomial ($p-1$ power) growth with respect to $u(t,x), Du(t,x)$ and b may be quickly increasing in $u(t,x)$.

1. **Existence in** $[0,T_0]$

Let $\Omega \subset R^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) and $V = W_0^{1,p}(\Omega)$ the usual Sobolev space of real valued functions which is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\| u \| = \left[\int_\Omega (|Du|^p + |u|^p)\right]^{1/p}.$$

Denote by $L^p(0,T_0;V)$ the Banach space of the set of measurable functions $u : (0,T_0) \to V$ such that $\| u \|^p$ is integrable and define the norm by

$$\| u \|^p_{L^p(0,T_0;V)} = \int_0^{T_0} \| u(t) \|^p_V \, dt.$$

The dual space of $L^p(0,T_0;V)$ is $L^q(0,T_0;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see, e.g., [6], [11]).

Assume that

I. The functions $a_i : Q_T \times R^{n+1} \times L^p(0,T_0;V) \to R$ satisfy the Carathéodory conditions for arbitrary fixed $v \in L^p(0,T_0;V)$ ($i = 0,1,...,n$).

II. There exist bounded (nonlinear) operators $g_1 : L^p(0,T_0;V) \to R^+ =$ and $k_1 : L^p(0,T_0;V) \to L^q(Q_{T_0})$ such that

$$|a_i(t,x,\zeta_0,\zeta;v)| \leq g_1(v)[|\zeta_0|^{p-1} + |\zeta|^{p-1}] + [k_1(v)](t,x)$$

for a.e. $(t,x) \in Q_{T_0}$, each $(\zeta_0,\zeta) \in R^{n+1}$ and $v \in L^p(0,T_0;V)$.

III. $\sum_{i=1}^n [a_i(t,x,\zeta_0,\zeta;v) - a_i(t,x,\zeta_0,\zeta^\star;v)](\zeta_i - \zeta_i^\star) > 0$ if $\zeta \neq \zeta^\star$.

IV. There exist bounded operators $g_2 : L^p(0,T_0;V) \to R^+$, $k_2 : L^p(0,T_0;V) \to L^1(Q_{T_0})$ such that

$$\sum_{i=0}^n a_i(t,x,\zeta_0,\zeta;v)\zeta_i \geq g_2(v)[|\zeta_0|^p + |\zeta|^p] - [k_2(v)](t,x)$$

for a.e. $(t,x) \in Q_{T_0}$, all $(\zeta_0,\zeta) \in R^{n+1}$, $v \in L^p(0,T_0;V)$ and $g_2(v) \geq c_2$ with some constant $c_2 > 0$,

$$\lim_{\|v\|_X \to \infty} \frac{\| k_2(v) \|_{L^1(Q_{T_0})}}{\| v \|_X^p} = 0 \tag{1.3}$$

where we used the notation $X = L^p(0,T_0;V)$. Further, if the sequence $(v_k)$is bounded in $L^p(0,T_0;V)$ and convergent in $L^p(Q_{T_0})$ then the sequence $[k_2(v_k)](t,x)$is equiintegrable in Q_{T_0}.

V. If $(u_k) \to u$ weakly in $L^p(0,T_0;V)$ and strongly in $L^p(Q_{T_0})$ then

$$\lim_{k\to\infty} \| a_i(t,x,u_k(t,x),Du_k(t,x);u_k) - a_i(t,x,u_k(t,x),Du_k(t,x);u) \|_{L^q(Q_{T_0})} = 0.$$

VI. $b : Q_{T_0} \times R \times L^p(0,T_0;V)$ satisfies the Carathéodory condition for each fixed $v \in L^P(0,T_0;V)$,

$$0 \leq b(t,x,\zeta_0;v)\zeta_0 \leq \psi(\zeta_0)\zeta_0 \leq \text{const}[b(t,x,\zeta_0;v)\zeta_0 + 1]$$

with some continuous nondecreasing function ψ with $\psi(0) = 0$.

VII. If $(u_k) \to u$ in the norm of $L^p(Q_{T_0})$ then for a suitable subsequence

$$b(t,x,u_k(t,x);u_k) \to b(t,x,u(t,x);u) \text{ for a.e. } (t,x) \in Q_{T_0}.$$

Theorem 1.1. *Assume I - VII. Then for any $f \in L^q(0,T_0;V^\star)$ there exists*

$$u \in L^p(0,T_0;V) \cap C([0,T_0];L^2(\Omega)) \textit{ such that } u(0) = 0,$$

$$b(t,x,u(t,x);u), \quad u(t,x)b(t,x,u(t,x);u) \in L^1(Q_{T_0}),$$

u is a distributional solution of (0.2). Further, for arbitrary $T \in [0, T_0]$,

$$v \in L^p(0, T_0; V) \cap C^1([0, T_0]; L^2(\Omega)) \text{ with } v(0) = 0, \quad v \in L^\infty(Q_{T_0})$$

we have

$$\int_0^T \langle D_t v(t), u(t) - v(t)\rangle dt + \tag{1.4}$$

$$\int_{Q_T} \left[\sum_{i=1}^n a_i(t, x, u, Du; u)(D_i u - D_i v) + a_0(t, x, u, Du; u)(u - v)\right] dtdx +$$

$$\frac{1}{2} \parallel u(T) - v(T) \parallel^2_{L^2(\Omega)} + \int_{Q_T} b(t, x, u(t, x); u)(u - v) dtdx =$$

$$\int_0^T \langle f(t), u(t) - v(t)\rangle dt.$$

Proof. Define

$$b_k(t, x, \zeta_0; v) = b(t, x, \zeta_0; v) \text{ if } b(t, x, \zeta_0; v) < k,$$

$$b_k(t, x, \zeta_0; v) = k \text{ if } b(t, x, \zeta_0; v) \geq k,$$

$$b_k(t, x, \zeta_0; v) = -k \text{ if } b(t, x, \zeta_0; v) \leq -k,$$

$$[A(u), v]_T =$$

$$\int_{Q_T} \left[\sum_{i=1}^n a_i(t, x, u(t, x), Du(t, x); u) D_i v + a_0(t, x, u(t, x), Du(t, x); u) v\right] dtdx,$$

$$[B_k(u), v]_T = \int_{Q_T} b_k(t, x, u(t, x); u) v dtdx, \quad u, v \in X = L^p(0, T_0; V);$$

with a fixed $u_0 \in X$

$$[\tilde{A}_{u_0}(u), v]_T =$$

$$\int_{Q_T} \left[\sum_{i=1}^n a_i(t, x, u(t, x), Du(t, x); u_0) D_i v + a_0(t, x, u(t, x), Du(t, x); u_0) v\right] dtdx.$$

It is not difficult to show that by I, II, IV (for fixed k)

$$(A + B_k) : L^p(0, T_0; V) \to L^q(0, T_0; V^\star)$$

is bounded (i.e. it maps bounded sets into bounded sets) and coercive, i.e.

$$\lim_{\|v\|_X \to \infty} \frac{[(A + B_k)(v), v]_{T_0}}{\parallel v \parallel_X} = +\infty.$$

Further, it is well known (see, e.g., [2]) that $\tilde{A}_{u_0} : X \to X^\star$ is demicontinuous (i.e. if $(u_j) \to u$ strongly in X then $(\tilde{A}_{u_0}(u_j)) \to \tilde{A}_{u_0}(u)$ weakly in $X^\star$) and pseudomonotone with respect to

$$D(L) = \{v \in X : D_t v \in X^\star, \quad v(0) = 0\},$$

i.e. if

$$(u_j) \to u \text{ weakly in } X, \quad (D_t u_j) \to D_t u \text{ weakly in } X^\star \text{ and}$$

$$\limsup_{j\to\infty} [\tilde{A}_{u_0}(u_j), u_j - u]_{T_0} \leq 0$$

then

$$\lim_{j\to\infty} [\tilde{A}_{u_0}(u_j), u_j - u]_{T_0} = 0 \text{ and } (\tilde{A}_{u_0}(u_j)) \to \tilde{A}_{u_0}(u) \text{ weakly in } X^\star.$$

By using assumption V, it is easy to show that also $A + B_k : X \to X^\star$ is demicontinuous and pseudomonotone with respect to $D(L)$ (see [10]).

Consequently, for each k there exists $u_k \in D(L)$ such that

$$D_t u_k + (A + B_k)(u_k) = f \text{ in } [0, T_0]. \tag{1.5}$$

(See, e.g., [2].) Applying (1.5) to $v = u_k$, we obtain by IV and Hölder's inequality for any $T \in [0, T_0]$

$$\frac{1}{2} \parallel u_k(T) \parallel^2_{L^2(\Omega)} + c_2 \parallel u_k \parallel^p_{L^p(0,T;V)} - \int_{Q_T} k_2(u_k) dt dx + \tag{1.6}$$

$$[B_k(u_k), u_k]_T \leq \parallel f \parallel_{L^q(0,T;V^\star)} \parallel u_k \parallel_{L^p(0,T;V)} .$$

According to VI $[B_k(u_k), u_k]_T \geq 0$, thus (1.3), (1.6), II imply that

$$\parallel u_k \parallel_{L^p(0,T_0;V)}, \quad \parallel A(u_k) \parallel^p_{L^p(0,T_0;V^\star)}, \quad [B_k(u_k), u_k]_{T_0} \text{ are bounded.} \tag{1.7}$$

Consequently, (1.6) and boundedness of k_2 imply that

$$\parallel u_k \parallel_{L^\infty(0,T_0;L^2(\Omega))} \text{ is bounded.} \tag{1.8}$$

By using VI, $|b_k| \leq |b| \leq |\psi|$, we find

$$|b_k(t, x, u_k(t, x); u_k)| \leq [\psi(1) + \psi(-1)| + b_k(t, x, u_k(t, x); u_k) u_k$$

which implies by (1.7) that

$$\int_{Q_{T_0}} |b_k(t,x,u_k(t,x);u_k)|dtdx \text{ is bounded.} \tag{1.9}$$

According to (1.5)

$$D_t u_k = [f - A(u_k)] - B_k(u_k)) \tag{1.10}$$

where the first term is bounded in $L^q(0,T;V^\star)$ and the second term is bounded in $L^1(Q_{T_0})$. Thus Proposition 1 of [3] implies that there is a subsequence of (u_k) (for simplicity denoted again by (u_k)) such that

$$(u_k) \to u \text{ weakly in } L^p(0,T_0;V), \text{ strongly in } L^p(Q_{T_0}) \text{ and a.e. in } Q_{T_0}. \tag{1.11}$$

Further, by (1.7) there exists $w \in L^q(0,T_0;V^\star)$ such that

$$(A(u_k)) \to w \text{ weakly in } L^q(0,T_0;V^\star). \tag{1.12}$$

Since by IV $k_2(u_k)(t,x)$ is equiintegrable in Q_{T_0}, we obtain from (1.6), (1.8), (1.11)

$$u \in L^\infty(0,T_0;L^2(\Omega)), \quad \lim_{T\to 0} \| u \|_{L^\infty(0,T_0;L^2(\Omega))} = 0. \tag{1.13}$$

We obtain from (1.11), assumption VII and the definition of b_k that

$$b_k(t,x,u_k(t,x);u_k) \to b(t,x,u(t,x);u) \text{ a.e. in } Q_{T_0}, \text{ so} \tag{1.14}$$

$$u_k b_k(t,x,u_k(t,x);u_k) \geq 0, \tag{1.15}$$

(1.7), Fatou's lemma imply

$$ub(t,x,u(t,x);u) \in L^1(Q_{T_0}) \quad \text{and so by VI} \quad u\psi(u) \in L^1(Q_{T_0}). \tag{1.16}$$

From (1.14), (1.16), VI and Vitali's theorem we obtain

$$b_k(t,x,u_k(t,x);u_k) \to b(t,x,u(t,x);u) \text{ in } L^1(Q_{T_0}), \quad \psi(u) \in L^1(Q_{T_0}) \tag{1.17}$$

because for arbitrary $\varepsilon > 0$

$$|b_k(t,x,\zeta_0;u_k)| \leq |b(t,x,\zeta_0;u_k)| \leq |\psi(\zeta_0)| \leq \varepsilon\psi(\zeta_0)\zeta_0 + \psi(1/\varepsilon) + |\psi(-1/\varepsilon)|$$

if $|\zeta_0| > 1/\varepsilon$, so by (1.7) $(b_k(t,x,u_k(t,x);u_k))$ is equiintegrable in Q_{T_0}.

From (1.5), (1.11), (1.12), (1.17) we obtain as $k \to \infty$

$$D_t u + w + b(t, x, u(t, x); u) = f \tag{1.18}$$

in distributional sense.

In order to show $w = A(u)$, we prove

$$\limsup_{k\to\infty} [A(u_k), u_k - u]_{T_0} \le 0. \tag{1.19}$$

Since by (1.11), V

$$\lim_{k\to\infty} [A(u_k) - \tilde{A}_u(u_k), u_k - u]_{T_0} = 0,$$

(1.19) will imply

$$\limsup_{k\to\infty} [\tilde{A}_u(u_k), u_k - u]_{T_0} \le 0,$$

thus we obtain from (1.11), (1.12) $w = \tilde{A}_u(u) = A(u)$ (see, e.g., Remark 4 in [8]).

Applying (1.5) to $u_k - v$ with some

$$v \in L^p(0, T_0; V) \cap C^1([0, T_0]; L^2(\Omega)) \cap L^\infty(Q_{T_0}) \text{ with } v(0) = 0,$$

we have for any $T \in [0, T_0]$

$$\int_0^T \langle D_t v, u_k - v\rangle dt + \frac{1}{2} \parallel u_k(T) - v(T) \parallel^2_{L^2(\Omega)} + \int_0^T \langle A(u_k), u_k - v\rangle dt + \tag{1.20}$$

$$\int_{Q_T} b_k(t, x, u_k(t, x); u_k)(u_k - v) dt dx = \int_0^T \langle f(t), u_k - v\rangle dt.$$

Since

$$[A(u_k), u_k - v]_T = [A(u_k), u_k - u]_T + [A(u_k), u - v]_T$$

and by Fatou's lemma, (1.7), (1.14), (1.15)

$$\liminf_{k\to\infty} \int_{Q_T} b_k(t, x, u_k(t, x); u_k) u_k dt dx \ge \int_{Q_T} b(t, x, u(t, x); u) u dt dx, \tag{1.21}$$

we obtain from (1.20) (by using (1.11), (1.12), (1.17))

$$\limsup_{k\to\infty} [A(u_k), u_k - u]_T \le \int_0^T \langle D_t v, v - u\rangle dt + \tag{1.22}$$

$$\int_{Q_T} b(t, x, u(t, x); u)(v - u) dt dx + \int_0^T \langle f(t) - w(t), u - v\rangle dt.$$

Consider the sequence (v_ν) of Theorem 3 in [3], approximating the function u which satisfies all the conditions of that theorem by (1.13), (1.17), and apply (1.22) to $v = v_\nu$. Then Proposition 3 of [3] implies (as $\nu \to \infty$) (1.19). Thus we have also

$$\lim_{k\to\infty} [A(u_k), u_k - u]_T = 0, \quad (A(u_k)) \to A(u) \text{ weakly in } L^q(0, T_0; V^\star) \tag{1.23}$$

(see, e.g., [8]). So, (1.18), $w = A(u)$ imply that u satisfies (0.2) in distributional sense.

Finally, we show $u \in C([0, T_0]; L^2(\Omega))$, $u(0) = 0$ and (1.4). From (1.11), (1.17), (1.20), (1.23) one obtains as $k \to \infty$

$$\limsup_{k\to\infty} \int_{Q_T} b_k(t, x, u_k(t, x); u_k) u_k dtdx \le \int_0^T \langle D_t v, v - u\rangle dt + \tag{1.24}$$

$$\int_{Q_T} b(t, x, u(t, x); u) v dtdx + [f - A(u), u - v]_T.$$

Applying (1.24) again to $v = v_\nu$ (approximating u), we find

$$\limsup_{k\to\infty} \int_{Q_T} b_k(t, x, u_k(t, x); u_k) u_k dtdx \le \int_{Q_T} b(t, x, u(t, x); u) u dtdx. \tag{1.25}$$

Further, by (1.11) for a.e. $T \in [0, T_0]$

$$(u_k(T)) \to u(T) \text{ a.e. in } \Omega,$$

so by (1.8) for a.e. $T \in [0, T_0]$

$$(u_k(T)) \to u(T) \text{ in } L^2(\Omega).$$

Consequently, from (1.20), (1.21), (1.25) one derives (1.4) for a.e. $T \in [0, T_0]$. Since all the terms in (1.4) are continuous in T, except possibly the term

$$\| u(T) - v(T) \|_{L^2(\Omega)}, \tag{1.26}$$

the latter can be extended to a continuous function in T and (1.4) holds for all $T \in [0, T_0]$.

For any smooth testing function w (defined in Ω) $(u(T), w)_{L^2(\Omega)}$ is continuous in T because (0.2) holds in distributional sense and the term in (1.26) is continuous in T, thus $u \in C([0, T_0]; L^2(\Omega))$ and so by (1.13) the initial condition $u(0) = 0$ is satisfied which completes the proof of Theorem 1.1.

2. **Boundedness and stabilization**

Denote by $L^p_{loc}(0,\infty;V)$ the set of functions $v:(0,\infty)\to V$ such that for each fixed finite $T_0>0$, $v|_{(0,T_0)}\in L^p(0,T_0;V)$ and let $Q_\infty=(0,\infty)\times\Omega$, $L^\alpha_{loc}(Q_\infty)$ the set of functions $v:Q_\infty\to R$ such that $v|_{Q_{T_0}}\in L^\alpha(Q_{T_0})$ for any finite T_0. By using a "diagonal process", it is not difficult to prove (see, e.g., [9])

Theorem 2.1. *Assume that we have functions* $a_i:Q_\infty\times R^{n+1}\times L^p_{loc}(0,\infty;V)\to R$, $b:Q_\infty\times R\times L^p_{loc}(0,\infty;V)\to R$ *such that they satisfy I - VII for any finite* $T_0>0$ *and* $a_i(t,x,\zeta_0,\zeta;v)|_{Q_{T_0}}$, $b(t,x,\zeta_0;v)|_{Q_{T_0}}$ *depend only on* $v|_{(0,T_0)}$ *(Volterra property). Then for any* $f\in L^q_{loc}(0,\infty;V^\star)$ *there exists* $u\in L^p_{loc}(0,\infty;V)$ *which is a solution for any finite* T_0 *(in the sense of Theorem 1.1).*

Theorem 2.2. *Let the assumptions of Theorem 2.1 be satisfied such that in IV we have* $g_2:L^p_{loc}(0,\infty;V)\to R^+$ *and* $k_2:L^p_{loc}(0,\infty;V)\to L^1_{loc}(Q_\infty)$, *satisfying for any* $v\in L^p_{loc}(0,\infty;V)$, $g_2(v)\ge c_2>0$ *and*

$$\int_\Omega |k_2(v)|dx\le c_4\left[\sup_{[0,t]}|y|^{p_1/2}+\varphi(t)\sup_{[0,t]}|y|^{p/2}+1\right]$$

with some constants c_4, $p_1<p$, $p>2$ *and* $\lim_\infty\varphi=0$ *where*

$$y(t)=\int_\Omega v(t,x)^2dx;$$

finally, $\| f(t)\|_{V^\star}$ *is bounded.*

Then for the solutions u, *formulated in Theorem 2.1,* $\int_\Omega u(t,x)^2dx$ *is bounded for* $t\in[0,\infty)$.

The idea of the proof. If u is a solution in $(0,\infty)$ then the assumptions of the theorem imply that $y(t)=\int_\Omega u(t,x)^2dx$ satisfies the inequality

$$y(T_2)-y(T_1)+c_5\int_{T_1}^{T_2}[y(t)]^{p/2}dt\le$$

$$c_6\int_{T_1}^{T_2}\left[\sup_{[0,t]}y^{p_1/2}+\varphi(t)\sup_{[0,t]}y^{p/2}+1\right]dt,\quad 0<T_1<T_2<\infty$$

with some constants $c_5>0, c_6$. It is not difficult to show that this inequality and $p>2$, $p_1<p$ imply the boundedness of y.

3. Examples

1. The conditions of Theorem 1.1 are satisfied if

$$a_i(t,x,\zeta_0,\zeta;v) = [H(v)](t,x)a_i^1(t,x,\zeta_0,\zeta) + [G(v)](t,x)a_i^2(t,x,\zeta_0,\zeta), \quad i = 1,...,n,$$

$$a_0(t,x,\zeta_0,\zeta;v) = [H(v)](t,x)a_0^1(t,x,\zeta_0,\zeta) + [G_0(v)](t,x)a_0^2(t,x,\zeta_0,\zeta)$$

where $H : L^p(Q_{T_0}) \to L^\infty(Q_{T_0})$ is bounded and continuous operator with the property: There exists a constant $c_2 > 0$ such that $H(v) \geq c_2$ for all v;

$$G, G_0 : L^p(Q_{T_0}) \to L^{\frac{p}{p-1-\rho}}(Q_{T_0}), \quad (0 \leq \rho < p-1)$$

are bounded and continuous operators, $G(v) \geq 0$ for all v and

$$\lim_{\|v\|_X \to \infty} \frac{\int_{Q_{T_0}} |G_0(v)|^{\frac{p}{p-1-\rho}}}{\| v \|_X^p} = 0.$$

Further, a_i^1, a_i^2 satisfy the usual conditions: They are Carathéodory functions,

$$|a_i^1(t,x,\zeta_0,\zeta)| \leq c_1(|\zeta_0|^{p-1} + |\zeta|^{p-1}) + k_1(x)$$

with some constant c_1, $k_1 \in L^q(\Omega)$, $i = 0,1,...,n$;

$$\sum_{i=1}^n [a_i^1(t,x,\zeta_0,\zeta) - a_i^1(t,x,\zeta_0,\zeta^\star)](\zeta_i - \zeta_i^\star) > 0 \text{ if } \zeta \neq \zeta^\star;$$

$$\sum_{i=0}^n a_i^1(t,x,\zeta_0,\zeta)\zeta_i \geq c_3(|\zeta_0|^p + |\zeta|^p) - k_2(x)$$

with some constant $c_3 > 0$, $k_2 \in L^1(\Omega)$;

$$|a_i^2(t,x,\zeta_0,\zeta)| \leq c_1(|\zeta_0|^\rho + |\zeta|^\rho), \quad 0 \leq \rho < p-1, \quad i = 0,1,...,n;$$

$$\sum_{i=1}^n [a_i^2(t,x,\zeta_0,\zeta) - a_i^2(t,x,\zeta_0,\zeta^\star)](\zeta_i - \zeta_i^\star) \geq 0;$$

$$\sum_{i=1}^n a_i^2(t,x,\zeta_0,\zeta)\zeta_i \geq 0.$$

By using Young's and Hölder's inequalities it is not difficult to show that the conditions I - V are fulfilled.

A simple special case for a_i^1, a_i^2 are:

$$a_i^1(t,x,\zeta_0,\zeta) = \zeta_i|\zeta|^{p-2}, \quad i = 1,...,n, \quad a_0^1(t,x,\zeta_0,\zeta) = \zeta_0|\zeta_0|^{p-2}, a_i^2 = 0.$$

The operator H may have e.g. one of the forms:

$\varphi\left(\int_{Q_t} bv\right)$ where $\varphi : R \to R$ is a continuous function, $\varphi \geq c_2 > 0$ (constant), $b \in L^q(Q_T)$;

$\varphi\left(\left[\int_{Q_t} |v|^\beta\right]^{1/\beta}\right)$ with some $1 \leq \beta \leq p$;

The operators G, G_0 may have e.g. one of the forms:

$$\psi_0\left(\int_0^t a(\tau,x)v(\tau,x)d\tau\right), \quad \psi_0\left(\int_\Omega a(t,x)v(t,x)dx\right),$$

$$\psi_0\left(\left[\int_0^t |v(\tau,x)|^\beta d\tau\right]^{\frac{1}{\beta}}\right),$$

where $\psi_0 : R \to R$ is continuous, $|\psi_0(\theta)| \leq \text{const}|\theta|^{p-1-\rho_0}$ with some $\rho_0 > \rho$, $\psi_0(\theta) \geq 0$ for G, $a \in L^\infty$.

The operators G, G_0 may have also the forms

$$\int_0^t h(t,\tau,x,v(\tau,x))d\tau \text{ or } h(t,x,v(\chi(t),x))$$

where

$$|h(t,\tau,x,\theta)|, \quad |h(t,x,\theta)| \leq \text{const}|\theta|^{p-1-\rho_0},$$

$0 \leq \chi(t) \leq t$, $\chi \in C^1$ and $h \geq 0$ for G.

2. The conditions on a_i of Theorem 1.1 are satisfied if

$$a_i(t,x,\zeta_0,\zeta;v) = [H_i(v)](t,x)\tilde{a}_i^1(t,x,\zeta_0,\zeta_i) + [G_i(v)](t,x)\tilde{a}_i^2(t,x,\zeta_0,\zeta_i)$$

where $\zeta_i \mapsto \tilde{a}_i^1(t,x,\zeta_0,\zeta_i)$ is strictly increasing for $i = 1,...,n$;

$$|\tilde{a}_i^1(t,x,\zeta_0,\zeta_i)| \leq c_1(|\zeta_0|^{p-1} + |\zeta_i|^{p-1}) + k_1(x)$$

with some constant c_1, $k_1 \in L^q(\Omega)$, $i = 0,1,...,n$;

$$\tilde{a}_i^1(t,x,\zeta_0,\zeta_i)\zeta_i \geq c_2|\zeta_i|^p - k_2(x), \quad i = 1,...,n$$

with some constant $c_2 > 0$, $k_2 \in L^1(\Omega)$; $\zeta_i \mapsto \tilde{a}_i^2(t,x,\zeta_0,\zeta_i)$ is monotone nondecreasing such that $\tilde{a}_i^2(t,x,\zeta_0,\zeta_i) = 0$ if $\zeta_i = 0$ $(i = 1,...,n)$;

$$|\tilde{a}_i^2(t,x,\zeta_0,\zeta_i)| \leq c_1(|\zeta_0|^\rho + |\zeta_i|^\rho) \text{ with } 0 \leq \rho < p-1, \quad i = 0,1,...,n.$$

Operators H_i satisfy the same conditions as H in Example 1 and operators G_i satisfy the same conditions as G, G_0, respectively, in Example 1.

Example on *b.* $b(t,x,\zeta_0;v)=\psi(\zeta_0)\tilde{G}(v)$ where $\tilde{G}:L^p(Q_{T_0})\to L^\infty(Q_{T_0})$ is a continuous operator with the property

$$0<c_1\le\tilde{G}(v)\le c_2<\infty \text{ for any } v$$

with some constants c_1,c_2.

The conditions of Theorem 2.1 are fulfilled for the Examples 1,2 if

$$H,H_i:L^p_{loc}(Q_\infty)\to L^\infty(Q_\infty),\quad G,G_i:L^p_{loc}(Q_\infty)\to L^{\frac{p}{p-1-\rho}}(Q_\infty)$$

satisfy the above conditions for any finite T_0 and they have the Volterra property; further, $a_i^1,a_i^2,\tilde{a}_i^1,\tilde{a}_i^2$ satisfy the above conditions for any t.

The conditions of Theorem 2.2 are satisfied if the following additional condition is fulfilled:

$$\int_\Omega|G_0(v)|^{\frac{p}{p-1-\rho}}dx\le c_4\left[\sup_{[0,t]}|y|^{p_1/2}+\varphi(t)\sup_{[0,t]}|y|^{p/2}+1\right]$$

for any $v\in L^p_{loc}(0,\infty;V)$ with $y(t)=\int_\Omega v(t,x)^2dx$ and $\| f(t)\|_{V^\star}$ is bounded.

The operator G_0 may have e.g. one of the forms

$$\psi_0\left(\int_\Omega a(t,x)v(t,x)dx\right),\quad \psi_0\left(\left[\int_\Omega|a(t,x)||v(t,x)|^\beta dx\right]^{1/\beta}\right),$$

$$\varphi_0(t)\chi_0\left(\left[\int_\Omega|a(t,x)||v(t,x)|^2dx\right]^{1/2}\right)$$

where $1\le\beta\le2$, $a\in L^\infty$, $\psi_0,\varphi_0,\chi_0:R\to R$ are continuous,

$$|\psi_0(\theta)|\le\text{const}|\theta|^{p-1-\rho_0}\text{ with some }\rho_0>\rho,$$

$$|\chi_0(\theta)|\le\text{const}|\theta|^{p-1-\rho},\quad\lim_\infty\varphi_0=0.$$

References

[1] R. A. Adams, *Sobolev spaces*, Academic Press, New York - San Francisco - London, 1975.

[2] J. Berkovits, V. Mustonen, *Topological degreee for perturbations of linear maximal monotone mappings and applications to a class of parabolic problems*, Rend. Mat. Ser. VII, **12**, Roma (1992), 597-621.

[3] F. E. Browder, *Strongly nonlinear parabolic equations of higher order*, Atti Acc. Lincei, **77** (1986), 159-172.

[4] M. Chipot, L. Molinet, *Asymptotic behavior of some nonlocal diffusion problems*, Applicable Analysis, **80** (2001), 279-315.

[5] M. Chipot, B. Lovat, *Existence and uniqueness results for a class of nonlocal elliptic and parabolic problems, advances in quenching*, Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal., **8**, 35-51.

[6] J. L. Lions, *Quelques métodes de résolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[7] L. Simon, *Strongly nonlinear parabolic functional differential equations*, *Annales Univ. Sci. Budapest*, **37** (1994), 215-228.

[8] L. Simon, *On systems of strongly nonlinear parabolic functional differential equations*, **33** (1996), 135-151.

[9] L. Simon, *On nonlinear hyperbolic functional differential equations*, *Math. Nachr.*, **217** (2000), 175-186.

[10] L. Simon, *On quasilinear parabolic functional differential equations of general divergence form*, Proceedings of the Conference Function Spaces, Differential Operators and Nonlinear Analysis, Svratka, 2004, submitted.

[11] E. Zeidler, *Nonlinear functional analysis and its applications II A and II B*, Springer, 1990.

DEPARTMENT OF APPLIED ANALYSIS, L. EÖTVÖS UNIVERSITY OF BUDAPEST,
PÁZMÁNY PÉTER SÉTÁNY 1/C, H-1117 BUDAPEST, HUNGARY
E-mail address: `simonl@ludens.elte.hu`