STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 2, June 2004

SIMPLE SUFFICIENT CONDITIONS FOR UNIVALENCE

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Abstract. We study some integral operators and determine conditions for the univalence of these integral operators.

1. Introduction

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in \mathcal{C}; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$. We denote by S the class of the functions $f \in A$ which are univalent in U.

2. Preliminary results

We will need the following theorems and lemma.

Theorem 2.1[2]. Let α be a complex number, $Re\ \alpha > 0$, and $f \in A$. If

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1}$$

for all $z \in U$, then for any complex number $\beta, Re\ \beta \geq Re\ \alpha$ the function

$$F_\beta(z) = \left[\beta \int_o^z u^{\beta-1} f'(u) du \right]^{\frac{1}{\beta}} \tag{2}$$

is in the class S.

Teorem 2.2 [1]. If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold:

$$\left| \frac{g(\xi) - g(z)}{1 - \overline{g(z)} g(\xi)} \right| \leq \left| \frac{\xi - z}{1 - \overline{z}\xi} \right|, \tag{3}$$

Received by the editors: 04.01.2003.
2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. integral operator, univalence.

$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2} \tag{4}$$

The equalities hold only in the case $g(z) = \varepsilon \frac{z+u}{1+\bar{u}z}$, where $|\varepsilon| = 1$ and $|u| < 1$.

The Schwarz Lemma [1]. Let the analytic function $f(z)$ be regular in the unit circle $|\, z \,| < 1$ and let $f(0) = 0$. If, in $|z| < 1, |f(z)| \leq 1$ then

$$|f(z)| \leq |z|, \qquad |z| < 1 \tag{5}$$

where equality can hold only if $f(z) = Kz$ and $|K| = 1$.

3. **Main results**

Theorem 3.1 Let γ be a complex number, $Re\, \gamma \geq 1$ and $g \in A$.

If

$$|g(z)| \leq 1 \tag{6}$$

for all $z \in U$, then the function

$$G_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1} e^{g(u)} du\right]^{\frac{1}{\gamma}} \tag{7}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z e^{g(u)}\, du. \tag{8}$$

The function f is regular in U. We have

$$\left(1-|z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| = \left(1-|z|^2\right)\, |z|\, |g'(z)| \tag{9}$$

From (6) and Theorem 2.2 we obtain

$$|g'(z)| \leq \frac{1}{1-|z|^2} \tag{10}$$

From (9) and (10) we obtain

$$\left(1-|z|^2\right) \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{11}$$

for all $z \in U$. From (8) we obtain $f'(z) = e^{g(z)}$, then from (11) and Theorem 2.1 for $Re\, \alpha = 1$, it follows that the function G_γ is in the class S.

Theorem 3.2. Let γ be a complex number, $Re\gamma = a > 0$, and the function $g \in A$. If

$$|zg'(z)| \leq 1 \tag{12}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}, \tag{13}$$

then the function

$$T_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1} \left(e^{g(u)}\right)^\gamma du\right]^{\frac{1}{\gamma}} \tag{14}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[e^{g(u)}\right]^\gamma du. \tag{15}$$

The function

$$h(z) = \frac{1}{|\gamma|} \frac{zf''(z)}{f'(z)}, \tag{16}$$

where the constant $|\gamma|$ satisfies the inequality (13), is regular in U.

From (15) and (16) we obtain

$$h(z) = \frac{\gamma}{|\gamma|} zg'(z), \tag{17}$$

Using (12) and (17) we obtain

$$|h(z)| < 1 \tag{18}$$

for all $z \in U$. From (17) we have $h(0) = 0$ and applying the Schwarz - Lemma we get

$$|h(z)| \leq |z| \tag{19}$$

for all $z \in U$, and hence, we obtain

$$\frac{1 - |z|^{2a}}{a} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{|\gamma|}{a} \left(1 - |z|^{2a}\right) |z|. \tag{20}$$

Let us consider the function $Q : [0, 1] \to \mathcal{R},\ Q(x) = \left(1 - x^{2a}\right) x,\ \ x = |z|.$ We have

$$Q(x) \leq \frac{2a}{(2a+1)^{\frac{2a+1}{2a}}} \tag{21}$$

for all $x \in [0, 1]$. From (21), (20) and (13) we obtain

$$\frac{1 - |z|^{2a}}{a} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 \tag{22}$$

for all $z \in U$. Then, from (22) and Theorem 2.1 for $Re\alpha = a$ it follows that the function T_γ is in the class S.

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