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ORTHOGONAL BASIS IN SOBOLEV SPACE $H_0^1(a,b)$

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Abstract. It is the purpose of this work to use the method of double-orthogonal sequences of Bergmann [1] to find an orthogonal basis in the Sobolev space $H_0^1(a,b)$. The elements of the basis are the solutions of some eigenvalue boundary problems.

In practice arise real difficulties in the problem of finding a base in Hilbert spaces. In case of Sobolev spaces a polynomial base is ussualy chosen, but other difficulties appear. Some of them were avoided using the finite element method. We give here a method of elimination of these difficulties using Bergmann's method of double orthogonal sequences [1].

Let $(H, (\cdot,\cdot))$, $(V, <\cdot,\cdot>)$ be real, separable Hilbert spaces and denote by $\|\cdot\|$, $|\cdot|$ the corresponding norms, respectively. In what follows, we use the next result due to Bergmann [1]:

Theorem 1. *Assume that $H \subset V$ and the imbedding $H \hookrightarrow V$ is compact,*

$$|x| \leq c\,\|x\| \quad , \quad \forall\, x \in H,$$

for some positive constant c. Then there exist an increasing, unbounded sequence $(\lambda_n)_{n\geq 1}$ of positive real numbers and a sequence $(e_n)_{n\geq 1} \subset H$ which is orthogonal with respect to both inner products, i.e.

$$(e_m, e_n) = \lambda_n \delta_{mn} \quad , \quad < e_m, e_n >= \delta_{mn}\,, \tag{1}$$

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for all positive integers m, n. Moreover, $(e_n)_{n\geq 1}$ *is complete in H.*

We will give a method to find an orthogonal basis in H. In fact, the elements of the basis are the solutions of some optimization problems.

In this sense, denote by $v_1 \in H$ a solution of the problem

$$\sup\{|x| \ ; \ x \in H, \ \|x\| = 1\}.$$

If $v_1, v_2, ..., v_{n-1}$ are already defined, then $v_n \in H$ is chosen as a solution of the problem

$$\sup\{|x| \ ; \ x \in H, \ \|x\| = 1, \ (x, v_i) = 0, \ 1 \leq i \leq n-1\}.$$

Finally,

$$e_n = \frac{1}{|v_n|} \cdot v_n \ , \quad n \geq 1.$$

For proofs and more details, see [1], [5]. The norms $\|\cdot\|$ and $|\cdot|$ are equivalent on finite dimensional subspaces of H.

Indeed, on $H_n = \mathrm{sp}\{e_1, e_2, ..., e_n\}$, $n \geq 1$, we have

$$\frac{1}{c}|x| \leq \|x\| \leq \sqrt{\lambda_n} \cdot |x| \ , \quad \forall\, x \in H_n.$$

Remark that from (1), we can derive the equalities

$$(e_m, e_n) = \lambda_n < e_m, e_n > \ , \quad \forall\, m, n \geq 1.$$

Because of completness of the system $(e_n)_{n\geq 1}$, it follows that

$$(e_n, v) = \lambda_n < e_n, v > \ , \quad \forall\, n \geq 1, \ v \in H. \tag{2}$$

In consequence, the elements of the orthogonal basis $(e_n)_{n\geq 1}$ can be considered as the solutions of the eigenvalue problem (2). In fact, this is an useful method to find a basis in a real separable Hilbert space, as we can see below.

Let $a < b$ be real numbers. We say that $u \in L^2(a,b)$ has generalized derivative (in Sobolev sense) if there exists $g \in L^2(a,b)$ such that

$$\int_a^b u\phi' = -\int_a^b g\phi \ , \quad \forall\, \phi \in C_0^\infty(a,b).$$

g (unique with this property) is called the generalized derivative of u an denote $g = u'$.

The set of all functions $u \in L^2(a,b)$ with $u(a) = u(b) = 0$, having generalized derivative is denoted by $H_0^1(a,b)$.

$H_0^1(a,b)$ also called Sobolev space is a Hilbert space relative to the scalar product

$$(u,v) = \int_a^b uv + \int_a^b u'v' \quad , \quad u,v \in H_0^1(a,b).$$

Here u', v' deonte the generalized derivatives of u, respective v. The corresponding norm is

$$\|u\| = \left(\int_a^b u^2 + \int_a^b u'^2 \right)^{1/2} \quad , \quad u \in H_0^1(a,b).$$

Consider also the Hilbert space $L^2(a,b)$ endowed with the usual scalar product

$$< u,v >= \int_a^b uv \quad , \quad u,v \in L^2(a,b)$$

and the usual norm

$$|u| = \left(\int_a^b u^2 \right)^{1/2} \quad , \quad u \in L^2(a,b).$$

The imbedding

$$H_0^1(a,b) \hookrightarrow L^2(a,b)$$

is compact because

$$|u| \leq \|u\| \quad , \quad \forall\, u \in H_0^1(a,b).$$

In order to give a method to find an orthogonal basis in $H_0^1(a,b)$, we will use theorem 1. The eigenvalue problem (2) can be written as

$$\int_a^b e_n v + \int_a^b e_n' v' = \lambda_n \int_a^b e_n v \quad , \quad \forall\, v \in H_0^1(a,b),\ n \geq 1. \tag{3}$$

But $v(a) = v(b) = 0$, so

$$\int_a^b e_n' v' = -\int_a^b e_n'' v,$$

if e_n is twice derivable. Hence (3) is equivalent with

$$\int_a^b e_n v - \int_a^b e_n'' v = \lambda_n \int_a^b e_n v,$$

so

$$\int_a^b (e_n'' + (\lambda_n - 1)e_n)v = 0 \quad , \quad \forall\, v \in H_0^1(a,b).$$

We deduce that $(e_n)_{n\geq 1}$ are the eigenfunctions of the following boundary problem

$$\begin{cases} e'' + \lambda e = 0 \\ e(a) = e(b) = 0 \end{cases} , \tag{4}$$

with $\lambda > 0$. The nontrivial solutions of the second order linear equation $e'' + \lambda e = 0$ are

$$e(x) = p\cos\sqrt{\lambda}x + q\sin\sqrt{\lambda}x \quad , \quad x \in (a,b),$$

for reals p, q, with $p^2 + q^2 \neq 0$.

The boundary conditions can be written as

$$\begin{cases} p\cos\sqrt{\lambda}a + q\sin\sqrt{\lambda}a = 0 \\ p\cos\sqrt{\lambda}b + q\sin\sqrt{\lambda}b = 0 \end{cases} . \tag{5}$$

If for example $q \neq 0$, we derive

$$-\frac{p}{q} = \tan\sqrt{\lambda}a = \tan\sqrt{\lambda}b,$$

so

$$\sqrt{\lambda}b - \sqrt{\lambda}a = n\pi \Rightarrow \lambda_n = \frac{n^2\pi^2}{(b-a)^2} \ , \ \ n \in \mathbf{N},\ n \geq 1.$$

In conclusion,

$$e_n(x) = -q\tan\frac{n\pi a}{b-a}\cos\frac{n\pi x}{b-a} + q\sin\frac{n\pi x}{b-a} \quad , \quad x \in (a,b)$$

is orthogonal basis in $H_0^1(a,b)$.

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