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THE BETA APPROXIMATING OPERATORS OF FIRST KIND

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Abstract. We shall define a general linear transform from which we obtain as particular case the beta first kind transform:

$$\mathcal{B}_{p,q}f = \frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f(t^a)dt \qquad (*)$$

We consider here only the particular case $a = 1$.
We obtain several positive linear operators as a particular case of this beta first kind transform. We apply the transform $(*)$ to Bernstein's operator B_n and thus we obtain different generalizations of this operator.

1. Introduction

Many authors introduced and studied positive linear operators, using Euler's beta function of first kind: [1], [2], [4], [6], [7], [8], [11].

Euler's beta function of first kind is defined for $p > 0$, $q > 0$ by the following formula

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \qquad (1.1)$$

The beta transform of the function f is defined by the following formula

$$\mathcal{B}_{p,q}f = \frac{1}{B(p,q)}\int_0^1 t^{p-q}(1-t)^{q-1}p(t)dt$$

We shall define a more general linear transform of a function f from which we obtain as particular case the beta first-kind transform.

For $a, b \in \mathbb{R}$, we define the (a,b)-beta transform of a function f (see [6])

$$\mathcal{B}_{p,q}^{(a,b)}f = \frac{1}{B(p,q)}\int_0^1 t^{p-1}(1-t)^{q-1}f(t^a(1-t)^b)dt \qquad (1.2)$$

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where $B(\cdot,\cdot)$ is the beta function (1.1) and f is any real measurable function defined on $(0,\infty)$ such that $\mathcal{B}_{p,q}^{(a,b)}|f| < \infty$.

If we put in (1.2) $b = 0$ we obtain the first-kind transform of a function f

$$\mathcal{B}_{p,q}^{(a)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t^a) dt \tag{1.3}$$

where $B(\cdot,\cdot)$ is the beta function (1.1) and f is any real measurable function defined on $(0,\infty)$ such that $\mathcal{B}_{p,q}^{(a)}|f| < \infty$. Clearly $\mathcal{B}_{p,q}^{(a)}$ is a positive linear functional.

We shall consider here the particular cases $a = 1$ and $a = -1$.

2. The beta first kind transform. Case $a = 1$

We shall consider here the particular case $a = 1$

$$\mathcal{B}_{p,q} f = \mathcal{B}_{p,q}^{(1)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f(t) dt. \tag{2.1}$$

We need to state and prove:

Lemma 2.1. *The moment of order k of the functional $\mathcal{B}_{p,q}$ has the following value*

$$\mathcal{B}_{p,q} e_k = \frac{p(p+1)\dots(p+k-1)}{(p+q)\dots(p+q+k-1)} = \frac{(p)_k}{(p+q)_k} \tag{2.2}$$

Proof.

$$\mathcal{B}_{p,q} e_k = \frac{1}{B(p,q)} \int_0^1 t^{p+k-1}(1-t)^{q-1} dt = \frac{B(p+k,q)}{B(p,q)} \tag{2.3}$$

By using successively k times the relation

$$B(p+1,q) = \frac{p}{p+q} B(p,q)$$

we find the relation

$$B(p+k,q) = \frac{p(p+1)\dots(p+k-1)}{(p+q)\dots(p+q+k-1)} B(p,q)$$

By replacing it into (2.3) we obtain the desired results (2.2). □

Consequently we obtain

$$\mathcal{B}_{p,q} e_1 = \frac{p}{p+q}, \quad \mathcal{B}_{p,q} e_2 = \frac{p(p+1)}{(p+q)(p+q+1)} \tag{2.4}$$

We impose that $\mathcal{B}_{p,q}e_1 = e_1$, that is $\dfrac{p}{p+q} = x$, or $\dfrac{p}{x} = \dfrac{q}{1-x}$, $x \in (0,1)$, $p > 0$ and we obtain the following linear transform

$$(\mathcal{B}_p f)(x) = \frac{1}{B\left(p, \dfrac{1-x}{x}p\right)} \int_0^1 t^{p-1}(1-t)^{\frac{1-x}{x}p-1} f(t)dt \tag{2.5}$$

Lemma 2.2. *One has*

$$\mathcal{B}_p((t-x)^2; x) = \frac{x^2(1-x)}{p+x}.$$

Proof. It is obtained from (2.4) for $q = \dfrac{1-x}{x}p$, $p+q = \dfrac{p}{x}$.

$$(\mathcal{B}_p e_2)(x) = \frac{p(p+1)}{\dfrac{p}{x}\left(\dfrac{p}{x}+1\right)} = \frac{p(p+1)x^2}{p(p+x)} = x^2 + \frac{(p+1)x^2}{p+x} - x^2 =$$

$$= x^2 + x^2\frac{p+1-p-x}{p+x} = x^2 + \frac{x^2(1-x)}{p+x}$$

and

$$\mathcal{B}_p((t-x)^2, x) = \frac{x^2(1-x)}{p+x}. \ \square$$

Particular cases.

a) Let $\mathcal{B}_\alpha$ be the beta operator defined by

$$(\mathcal{B}_\alpha f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha}, \dfrac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} f(t)dt \tag{2.6}$$

$\alpha > 0$, $x \in (0,1)$. If f is defined on $[0,1]$ we set

$$(\mathcal{B}_\alpha f)(0) = f(0), \quad (\mathcal{B}_\alpha f)(1) = f(1).$$

The operator (2.6) has been considered by G. Mühlbach [7] and it is obtained by (2.5) if we choose in (2.5) $p = \dfrac{x}{\alpha}$.

Lemma 2.3. *One has*

$$\mathcal{B}_\alpha((t-x)^2, x) = \frac{\alpha}{1+\alpha}x(1-x).$$

Proof. $\mathcal{B}_\alpha e_2 = \dfrac{\dfrac{x}{\alpha}\left(\dfrac{x}{\alpha}+1\right)}{\dfrac{1}{\alpha}\left(\dfrac{1}{\alpha}+1\right)} = \dfrac{x(x+\alpha)}{1+\alpha} = x^2 + \left(\dfrac{x^2+\alpha x}{1+\alpha} - x^2\right) =$

$$= x^2 + \frac{\alpha x - \alpha x^2}{1+\alpha} = x^2 + \frac{\alpha}{1+\alpha}x(1-x) \ \Rightarrow$$

$$\mathcal{B}_\alpha(t-x)^2(x) = \frac{\alpha}{1+\alpha}x(1-x). \ \square$$

For $\alpha = 1/n$ we obtain $\mathcal{B}_{1/n}(t-x)^2(x) = \dfrac{x(1-x)}{n}$.

A slight modification of $\mathcal{B}_\alpha$ is the operator $\mathcal{B}_\alpha^*$ given by

$$(\mathcal{B}_\alpha^* f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha}+1; \dfrac{1-x}{\alpha}+1\right)} \int_0^1 t^{\frac{x}{\alpha}}(1-t)^{\frac{1-\alpha}{\alpha}} f(t)dt, \tag{2.7}$$

$\alpha > 0$, $x \in [0,1]$, which, for $\alpha = 1/n$, $n \in \mathbb{N}$, has been introduced by A. Lupaş [4] and it is obtain by (2.5) if we replace in (2.5) $p = nx + 1$.

A significant difference between $\mathcal{B}_\alpha$ and $\mathcal{B}_\alpha^*$ is that $\mathcal{B}_\alpha$ reproduces linear functions whereas $\mathcal{B}_\alpha^*$ does not.

b) Another beta first-kind operator it is obtained by (2.5) for $p = \dfrac{x}{\alpha(1-x)}$.

$$(\overline{\mathcal{B}}_\alpha f)(x) = \frac{1}{B\left(\dfrac{x}{\alpha(1-x)}; \dfrac{1}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha(1-x)}-1}(1-t)^{\frac{1}{\alpha}-1} f(t)dt, \tag{2.8}$$

$\alpha > 0$, $x \in (0,1)$, where f is any real measurable function defined on (0,1) such that $(\overline{B}_\alpha|f|)(x) < \infty$. The operator (2.7) was introduced by S. Rathore [8] for $\alpha = 1/n$, $n \in \mathbb{N}$.

Lemma 2.4. *One has*

$$\overline{\mathcal{B}}_\alpha((t-x)^2; x) = \frac{\alpha x(1-x)^2}{1+\alpha(1-x)}.$$

Proof. $\overline{\mathcal{B}}_\alpha e_2 = \dfrac{\dfrac{x}{\alpha(1-x)}\left(\dfrac{x}{\alpha(1-x)}+1\right)}{\dfrac{1}{\alpha(1-x)}\left(\dfrac{1}{\alpha(1-x)}+1\right)} = \dfrac{x(x+\alpha(1-x))}{1+\alpha(1-x)} =$

$$= x^2 + \left(\frac{x^2+\alpha x(1-x)}{1+\alpha(1-x)} - x^2\right) = x^2 + \frac{\alpha x(1-x)^2}{1+\alpha(1-x)} \ \Rightarrow$$

$$\overline{\mathcal{B}}_\alpha (t-x)^2(x) = \frac{\alpha x(1-x)^2}{1+\alpha(1-x)}. \ \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{\mathcal{B}}_{1/n}(t-x)^2(x) = \frac{x(1-x)^2}{n+1-x}$$

c) Let $\widetilde{\mathcal{B}}_\alpha$ be the operator defined by

$$(\widetilde{\mathcal{B}}_\alpha f)(x) = \frac{1}{B\left(\frac{1}{\alpha}, \frac{1-x}{\alpha x}\right)} \int_0^1 t^{\frac{1}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha x}-1} f(t)dt \tag{2.9}$$

$\alpha > 0$, $x \in (0,1)$. The operator (2.8) is obtained by (2.5) if we choose in (2.5) $p = \dfrac{1}{\alpha}$.

Lemma 2.5. *One has*

$$\widetilde{\mathcal{B}}_\alpha((t-x)^2; x) = \frac{\alpha x^2(1-x)}{1+\alpha x}$$

Proof. $\widetilde{\mathcal{B}}_\alpha e_2 = \dfrac{\frac{1}{\alpha}\left(\frac{1}{\alpha}+1\right)}{\frac{1}{\alpha x}\left(\frac{1}{\alpha x}+1\right)} = \dfrac{\alpha+1}{\alpha^2}\dfrac{\alpha^2 x^2}{1+\alpha x} = \dfrac{\alpha+1}{1+\alpha x}x^2 =$

$$= x^2 + \left(\frac{\alpha+1}{1+\alpha x}x^2 - x^2\right) = x^2 + \frac{\alpha x^2 - \alpha x^3}{1+\alpha x} = x^2 + \frac{\alpha x^2(1-x)}{1+\alpha x} \ \Rightarrow$$

$$\widetilde{\mathcal{B}}_\alpha(t-x)^2(x) = \frac{\alpha x^2(1-x)}{1+\alpha x}. \ \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\widetilde{\mathcal{B}}_{1/n}(t-x)^2 = \frac{x^2(1-x)}{n+x}.$$

3. The functional $P_n^{(p,q)} f = \mathcal{B}_{p,q}(B_n f)$

Now let us apply the transform (2.1) to the Bernstein operator B_n, defined by [3]

$$(B_n f)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right)$$

We may state and prove

Theorem 3.1. *The $\mathcal{B}_{p,q}$ transform of $B_n f$ can be expressed under the following form*

$$P_n^{(p,q)} f = \mathcal{B}_{p,q}(B_n f) = \sum_{k=0}^{n} \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} f\left(\frac{k}{n}\right)$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{p(p+1)\dots(p+k-1)q(q+1)\dots(q+n-k-1)}{(p+q)(p+q+1)\dots(p+q+n-1)} f\left(\frac{k}{n}\right) \tag{3.1}$$

Proof.

$$P_n^{(p,q)} f = \mathcal{B}_{p,q}(B_n f) = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{B(p,q)} \int_0^1 t^{p+k-1}(1-t)^{q+n-k+1} dt \cdot f\left(\frac{k}{n}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{B(p+k, q+n-k)}{B(p,q)} f\left(\frac{k}{n}\right) =$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{p(p+1)\dots(p+k-1)q(q+1)\dots(q+n-k-1)}{(p+q)(p+q+1)\dots(p+q+n-1)} f\left(\frac{k}{n}\right).$$

Theorem 3.2. *One has*

$$P_n^{(p,q)} e_1 = \mathcal{B}_{p,q}(B_n e_1) = \frac{p}{p+q}$$

$$P_n^{(p,q)} e_2 = \mathcal{B}_{p,q}(B_n e_2) = \frac{p}{(p+q)(p+q+1)} \left(p+1+\frac{q}{n}\right) \tag{3.2}$$

Proof. $P_n^{(p,q)} e_1 = \mathcal{B}_{pq}(B_n e_1) = \dfrac{1}{B(p,q)} \displaystyle\int_0^1 t^p (1-t)^{q-1} dt = \dfrac{B(p+1,q)}{B(p,q)} = \dfrac{p}{p+q}$.

$$P_n^{(p,q)} e_2 = \mathcal{B}_{p,q}(B_n e_2) = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} \left(t^2 + \frac{t(1-t)}{n}\right) dt =$$

$$= \frac{1}{B(p,q)} \left(\int_0^1 t^{p+1}(1-t)^{q-1} dt + \frac{1}{n} \int_0^1 t^p (1-t)^q dt \right) =$$

$$= \frac{B(p+2,q)}{B(p,q)} + \frac{1}{n} \frac{B(p+1,q+1)}{B(p,q)} = \frac{p(p+1)}{(p+q)(p+q+1)} + \frac{1}{n} \frac{pq}{(p+q)(p+q+1)}. \square$$

We impose that $P_n^{(p,q)} e_1 = \mathcal{B}_{p,q}(B_n e_1) = e_1$, that is $\dfrac{p}{p+q} = x$ or $q = \dfrac{1-x}{x} p$, $x \in (0,1)$, $p > 0$. We obtain from Theorem 3.1 and Theorem 3.2 the following results.

Corollary 3.3. *One has*

$$P_n^{(p)} f = \mathcal{B}_p(B_n f) = \sum_{k=0}^{n} v_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.3}$$

where

$$v_{n,k}(x) = \binom{n}{k} \frac{p(p+1)\dots(p+k-1)p(1-x)(p(1-x)+x)\dots(p(1-x)+(n-k-1)x)}{p(p+x)\dots(p+(n-1)x)} x^k.$$

Proof. If we put in (3.1) $q = p\dfrac{1-x}{x}$, then $p+q = \dfrac{p}{x}$ and we obtain

$$P_n^{(p)} f = \mathcal{B}_p(B_n f) = \sum_{k=0}^{n} v_{n,k}(x) f\left(\frac{k}{n}\right)$$

where

$$v_{n,k}(x) = \binom{n}{k} \frac{(p)_k (q)_{n-k}}{(p+q)_n} = \binom{n}{k} \frac{(p)_k \left(p\dfrac{1-x}{x}\right)_{n-k}}{\left(\dfrac{p}{x}\right)_n} =$$

$$= \binom{n}{k} \frac{p(p+1)\dots(p+k-1)\left(p\dfrac{1-x}{x}\right)\left(p\dfrac{1-x}{x}+1\right)\dots\left(p\dfrac{1-x}{x}+n-k-1\right)}{\dfrac{p}{x}\left(\dfrac{p}{x}+1\right)\dots\left(\dfrac{p}{x}+n-1\right)} =$$

$$= \binom{n}{k} \frac{p(p+1)\dots(p+k-1)p(1-x)(p(1-x)+1)\dots(p(1-x)+(n-k-1)x)}{p(p+x)\dots(p+(n-1)x)} x^k. \ \square$$

Corollary 3.4. *One has* $(P_n^{(p)} e_1)(x) = \mathcal{B}_p(B_n e_1)(x) = x$*;*

$$(P_n^{(p)} e_2)(x) = \mathcal{B}_p(B_n e_2)(x) = x^2 + x(1-x)\frac{nx+p}{n(x+p)};$$

$$P_n^{(p)}(t-x)^2(x) = \mathcal{B}_p(B_n(t-x)^2)(x) = \frac{x(1-x)}{n} \cdot \frac{nx+p}{x+p} \tag{3.4}$$

Proof. $(P_n^{(p)} e_1)(x) = \mathcal{B}_p(B_n e_1)(x) = \dfrac{p}{p+q} = \dfrac{px}{p} = x.$

$$(P_n^{(p)} e_2)(x) = \mathcal{B}_p(B_n e_2)(x) = \frac{p}{(p+q)(p+q+1)}\left(p+1+\frac{q}{n}\right) =$$

$$= \frac{p}{\dfrac{p}{x}\left(\dfrac{p}{x}+1\right)}\left(p+1+\frac{p}{n}\frac{1-x}{x}\right) = \frac{x^2}{p+x}\left(p+1+\frac{p}{n}\frac{1-x}{x}\right) =$$

$$= x^2 + \frac{x^2}{p+x}\left(p+1+\frac{p}{n}\frac{1-x}{x}\right) - x^2 = x^2 + x^2\left(\frac{p+1}{p+x} + \frac{p}{n}\frac{1-x}{x(p+x)} - 1\right) =$$

$$= x^2 + x^2 \frac{(1-x)(p+nx)}{nx(p+x)} = x^2 + \frac{x(1-x)}{n} \frac{nx+p}{x+p}.$$

$$P_n^{(p)}(t-x)^2(x) = \mathcal{B}_p(B_n(t-x)^2)(x) = \frac{x(1-x)}{n} \frac{nx+p}{x+p}. \square$$

Particular cases

a) If we put in (3.3) $p = \dfrac{x}{\alpha}$, $\alpha > 0$, we obtain

$$(P_n^{(\alpha)} f)(x) = \sum_{k=0}^{n} v_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.5}$$

$$v_{n,k}(x) = \binom{n}{k} \frac{x(x+\alpha)\ldots(x+(k-1)\alpha)(1-x)(1-x+\alpha)\ldots(1-x+(n-k-1)\alpha)}{(1+\alpha)(1+2\alpha)\ldots(1+(n-1)\alpha)}$$

This operator has been considered by D. D. Stancu [9], which, for $\alpha = 1/n$, $n \in \mathbb{N}$, has been introduced by L. Lupaş and A. Lupaş [5]:

$$(L_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(nx)_k (n(1-x))_{n-k}}{(n)_n} f\left(\frac{k}{n}\right) \tag{3.6}$$

Corollary 3.5. *One has* $P_n^{(\alpha)}(t-x)^2(x) = \dfrac{1+n\alpha}{n(1+\alpha)} x(1-x)$.

Remark. For $\alpha = 1/n$ we obtain

$$P_n(t-x)^2(x) = \frac{2}{n+1} x(1-x).$$

b) Another operator it is obtained by (3.3) for $p = \dfrac{x}{\alpha(1-x)}$, $\alpha > 0$.

$$(\overline{P}^{(\alpha)} f)(x) = \sum_{k=0}^{n} \overline{v}_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.7}$$

$$\overline{v}_{n,k}(x) = \binom{n}{k} \frac{x(x+\alpha(1-x))\ldots(x+(k-1)\alpha(1-x))(1+\alpha)\ldots(1+(n-k-1)\alpha)}{(1+\alpha)(1-x)\ldots(1+(n-1)\alpha(1-x))}.$$

Corollary 3.6. *One has*

$$\overline{P}_n^{(\alpha)}(t-x)^2(x) = \frac{x(1-x)}{n} \cdot \frac{1+n\alpha(1-x)}{1+\alpha(1-x)}$$

Remark. For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{P}_n(t-x)^2(x) = \frac{x(1-x)(2-x)}{n+1-x}.$$

c) Let $\widetilde{P}_n^{(\alpha)}$ be the operator defined by

$$(\widetilde{P}_n^{(\alpha)} f)(x) = \sum_{k=0}^{n} \widetilde{v}_{n,k}(x) f\left(\frac{k}{n}\right) \tag{3.8}$$

$$\widetilde{v}_{n,k}(x) = \binom{n}{k} \frac{(1+\alpha)\dots(1+(k-1)\alpha)(1-x)(1-x+\alpha x)\dots(1-x+(n-k-1)\alpha x)}{(1+\alpha x)(1+2\alpha x)\dots(1+(n-1)\alpha x)} x^{n-k}$$

$\alpha > 0$, $x \in (0,1)$. This operator is obtained by (3.3) for $p = 1/\alpha$, $\alpha > 0$.

Corollary 3.7. *One has*

$$\widetilde{P}_n^{(\alpha)}(t-x)^2(x) = \frac{x(1-x)}{n} \cdot \frac{1+n\alpha x}{1+\alpha x}$$

Remark. For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\widetilde{P}_n(t-x)^2(x) = \frac{x(1-x)(1+x)}{n+x}.$$

From the operators (3.5), (3.7) and (3.8), for $\alpha = 0$ we obtain the operator of S. N. Bernstein.

4. The beta first kind transform. Case $a = -1$

We consider now the case $a = -1$. If we put $a = -1$ in (1.3) we obtain

$$\mathbf{B}_{p,q} f = \mathcal{B}_{p,q}^{(-1)} f = \frac{1}{B(p,q)} \int_0^1 t^{p-1}(1-t)^{q-1} f\left(\frac{1}{t}\right) dt \tag{4.1}$$

Lemma 4.1. *The moment of order k $(1 \le k < p)$ of the functional $\mathbf{B}_{p,q}$ has the following value*

$$\mathbf{B}_{p,q} e_k = \frac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}, \quad 1 \le k < p \tag{4.2}$$

Proof.

$$\mathbf{B}_{p,q} e_k = \frac{1}{B(p,q)} \int_0^1 t^{p-k-1}(1-t)^{q-1} dt = \frac{B(p-k,q)}{B(p,q)} \tag{4.3}$$

By using successively k times the relation

$$B(p-1,q) = \frac{p+q-1}{p-1} B(p,q)$$

we find the relation

$$B(p-k,q)=\frac{(p+q-1)\dots(p+q-k)}{(p-1)\dots(p-k)}B(p,q)$$

By replacing it into (4.3) we obtain the desired results (4.2). □

Consequently we obtain

$$\mathbf{B}_{p,q}e_1=\frac{p+q-1}{p-1},\quad \mathcal{B}_{p,q}e_2=\frac{(p+q-1)(p+q-2)}{(p-1)(p-2)} \tag{4.4}$$

We impose that $B_{p,q}e_1=e_1$, that is $\frac{p+q-1}{p-1}=x$ or $p-1=\frac{q}{x-1}$ and we obtain the following linear transform, defined for $x>1$ and $p>2$:

$$\mathbf{B}_p f=\frac{1}{B(p,(p-1)(x-1))}\int_0^1 t^{p-1}(1-t)^{(p-1)(x-1)-1}f\left(\frac{1}{t}\right)dt \tag{4.5}$$

Lemma 4.2. *One has*

$$\mathbf{B}_p((t-x)^2;x)=\frac{x(x-1)}{p-2}$$

Proof. It is obtained from Lemma 2.1 for $q=(p-1)(x-1)$, $p+q=1+(p-1)x$ and

$$\mathbf{B}_p e_2(x)=\frac{(p-1)x((p-1)x-1)}{(p-1)(p-2)}=x^2+\frac{(p-1)x^2-x}{p-2}-x^2=$$

$$=x^2+\frac{(p-1)x^2-x-(p-2)x^2}{p-2}=x^2+\frac{x^2-x}{p-2}=x^2+\frac{x(x-1)}{p-2}$$

and

$$\mathbf{B}_p((t-x)^2;x)=\frac{x(x-1)}{p-2}.\ \square$$

Particular cases

a) Let $\mathbf{B}_\alpha$ be the beta operator defined by

$$(\mathbf{B}_\alpha f)(x)=\frac{1}{B\left(1+\frac{1}{\alpha},\frac{x-1}{\alpha}\right)}\int_0^1 t^{\frac{1}{\alpha}}(1-t)^{\frac{x-1}{\alpha}-1}f\left(\frac{1}{t}\right)dt \tag{4.6}$$

$\alpha\in(0,1)$, $x\in(1,\infty)$. If f is defined on $[1,\infty)$ we set $(\mathbf{B}_\alpha f)(1)=f(1)$.

This operator is obtained by (4.5) if we choose in (4.5) $p=1+\frac{1}{\alpha}$.

Lemma 4.3. *One has*

$$\mathbf{B}_\alpha((t-x)^2;x) = \frac{\alpha}{1-\alpha}x(x-1)$$

Proof. $\mathbf{B}_\alpha e_2 = \dfrac{\dfrac{x}{\alpha}\left(\dfrac{x}{\alpha}-1\right)}{\dfrac{1}{\alpha}\left(\dfrac{1}{\alpha}-1\right)} = \dfrac{x(x-\alpha)}{1-\alpha} = x^2 + \left(\dfrac{x^2-x\alpha}{1-\alpha} - x^2\right) =$

$$= x^2 + \frac{x^2 - x\alpha - x^2 + \alpha x^2}{1-\alpha} = x^2 + \frac{\alpha x(x-1)}{1-\alpha}$$

and

$$\mathbf{B}_\alpha((t-x)^2;x) = \frac{\alpha}{1-\alpha}(x-1). \;\square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\mathbf{B}_{\frac{1}{n}}((t-x)^2;x) = \frac{x(x-1)}{n-1}.$$

b) Let $\overline{\mathbf{B}}_\alpha$ be the operator defined by

$$(\overline{\mathbf{B}}_\alpha f)(x) = \frac{1}{B\left(1+\dfrac{1}{\alpha(x-1)}, \dfrac{1}{\alpha}\right)} \int_0^1 t^{\frac{1}{\alpha(x-1)}}(1-t)^{\frac{1}{\alpha}-1} f\left(\frac{1}{t}\right) dt \tag{4.7}$$

$\alpha \in (0,1)$, $x \in \left(1, 1+\dfrac{1}{\alpha}\right)$. This operator is obtained by (4.5) if we choose in (4.5) $p = 1 + \dfrac{1}{\alpha(x-1)}$.

Lemma 4.4. *One has*

$$\widetilde{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha(x-1)^2}{1-\alpha(x-1)}.$$

Proof. $\overline{\mathbf{B}}_\alpha e_2 = \dfrac{\dfrac{x}{\alpha(x-1)}\left(\dfrac{x}{\alpha(x-1)}-1\right)}{\dfrac{1}{\alpha(x-1)}\left(\dfrac{1}{\alpha(x-1)}-1\right)} = \dfrac{x(x-\alpha(x-1))}{1-\alpha(x-1)} =$

$$= x^2 + \frac{x^2-\alpha x(x-1)}{1-\alpha(x-1)} - x^2 = x^2 + \frac{x^2 - \alpha x(x-1) - x^2 + \alpha x^2(x-1)}{1-\alpha(x-1)} =$$

$$= x^2 + \frac{\alpha x(x-1)^2}{1-\alpha(x-1)}$$

and

$$\overline{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha x(x-1)^2}{1-\alpha(x-1)}. \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{\mathbf{B}}_{1/n}((t-x)^2;x) = \frac{x(x-1)^2}{n+1-x}.$$

c) Another beta first-kind operator it is obtained by (2.5) for $p = 1 + \dfrac{1}{\alpha x}$.

$$(\widetilde{\mathbf{B}}_\alpha f)(x) = \frac{1}{B\left(1+\dfrac{1}{\alpha x};\dfrac{x-1}{\alpha x}\right)} \int_0^1 t^{\frac{1}{\alpha x}}(1-t)^{\frac{x-1}{\alpha x}-1} f\left(\frac{1}{t}\right) dt \tag{4.8}$$

$\alpha \in (0,1)$, $x \in (1, 1/\alpha)$, where f is any real measurable function defined on $(1, 1/\alpha)$, such that $(\overline{\mathbf{B}}_\alpha |f|)(x) < \infty$.

Lemma 4.5. *One has*

$$\widetilde{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha x^2(x-1)}{1-\alpha x}$$

Proof. $\widetilde{\mathbf{B}}_\alpha e_2 = \dfrac{\dfrac{1}{\alpha}\left(\dfrac{1}{\alpha}-1\right)}{\dfrac{1}{\alpha x}\left(\dfrac{1}{\alpha x}-1\right)} = \dfrac{1-\alpha}{1-\alpha x}x^2 = x^2 + \dfrac{1-\alpha}{1-\alpha x}x^2 - x^2 =$

$$= x^2 + \frac{x^2 - \alpha x^2 - x^2 + \alpha x^3}{1-\alpha x} = x^2 + \frac{\alpha x^2(x-1)}{1-\alpha x}$$

and

$$\widetilde{\mathbf{B}}_\alpha((t-x)^2;x) = \frac{\alpha x^2(x-1)}{1-\alpha x}. \square$$

For $\alpha = 1/n$, $n \in \mathbb{N}$, we obtain

$$\overline{\mathbf{B}}_{1/n}((t-x)^2;x) = \frac{x^2(x-1)}{n-x}.$$

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