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ON SOME PROPERTIES OF THE STARLIKE SETS AND GENERALIZED CONVEX FUNCTIONS. APPLICATION TO THE MATHEMATICAL PROGRAMMING WITH DISJUNCTIVE CONSTRAINTS

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Abstract. In this paper we give an extension for starlike sets of the well known property that any convex set in the n-dimensional Euclidian space is the convex hull of its extremal points. We establish some relationships between two classes of starlike functions and the convex and quasi-convex classes of functions. We consider also the concepts of marginal points and starlike hull of a given set, and we show that a starlike set is the starlike hull of its marginal point set. For the starlike quasi-convex mathematical programming with disjunctive constraints, we show the starlikeness property of its feasible set.

1. Introduction

The main goal of this paper is to give an extension for starlike sets of the well known property that any convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.

We consider also, in section 2, the classes of starlike convex and starlike quasi-convex functions and we present some relationships between them.

In Section 3, we introduce the concepts of marginal points and starlike hull of a given set, and we show that a starlike set is the starlike hull of its marginal points.

For the starlike quasi-convex mathematical programming with disjunctive constraints, in section 4, we prove the starlikeness property of its feasible set.

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2. **Preliminaries about starlike sets and functions**

We shall present some concepts and preliminaries properties concerning starlike sets and functions useful in order to obtain the main results in this paper.

Definition 1. *A set* $A \subseteq \mathbb{R}^n$*,* $A \neq \emptyset$ *is called convex, if* $\lambda x + (1-\lambda)y \in A$*,* $\forall x, y \in A$ *and* $\forall \lambda \in (0,1)$*.*

Let $[x, y] = \{\lambda x + (1-\lambda)\, y \in \mathbb{R}^n | \lambda \in [0,1]\}$ be the segment that links the points $x, y \in \mathbb{R}^n$.

The following property holds for any collection of convex sets in $\mathbb{R}^n$ (see, e.g. [12]).

Proposition 1. *The intersection of any collection of convex sets in* $\mathbb{R}^n$ *is a convex set.*

Definition 2. *Let* $A \subseteq \mathbb{R}^n, A \neq \emptyset$*. The convex hull of a set* A *is the intersection of all convex sets in* $\mathbb{R}^n$ *containing* A *and is denoted* $convA$*.*

Proposition 2. *If* $B \subseteq A \subseteq \mathbb{R}^n$ *and* A *is a convex set, then* $convB \subseteq A$*.*

Definition 3. *Let* $A \subseteq \mathbb{R}^n, A \neq \emptyset$*. One point* $x \in A$ *is called extreme point of* A*, if there exists no two distinct points* $x^{'}, x^{''} \in A$ *and* $\lambda \in (0,1)$ *so that* $x = \lambda x^{'} + (1-\lambda)x^{''}$*. Let denote by* $ext(A)$ *the set of all extreme points of* A*.*

The following fundamental result is known as Minkowski's Theorem (see, [1], [2], [11]).

Theorem 3. *Any convex and compact set in* $\mathbb{R}^n$ *is the convex hull of its extreme points.*

Definition 4. *(*[15],[4]*) Let* $A \subseteq \mathbb{R}^n$*,* $A \neq \emptyset$*. The set* A *is called starlike with respect to the point* $x^0 \in A$*, if* $\lambda x^0 + (1-\lambda)\, y \in A$*,* $\forall y \in A$ *and* $\lambda \in (0,1)$*. The point* $x^0 \in A$ *with the above property is said to be starlikeness center of the set* A*. A set* A *that posses at least one starlikeness center is said to be a starlike set.*

We mention that a characterization of starlike sets in term of their maximal convex subsets was given by Bragard [3].

From Definition 4 it results without difficulty the following two properties:

Theorem 4. *If* $A_1, A_2, ..., A_s$ *are starlike sets in* $\mathbb{R}^n$, *having a common starlike center, then* $\bigcup_{k=1}^{s} A_k$ *is a starlike set.*

Theorem 5. *If* $A_1, A_2, ..., A_s$ *are starlike sets in* $\mathbb{R}^n$, *having a common starlike center, then* $\bigcap_{k=1}^{s} A_k$ *is a starlike set.*

Definition 5. *(*[5]*) Let* $A \subseteq \mathbb{R}^n, A \neq \emptyset$. *The set of all points* $z \in \mathbb{R}^n$, *such that* $[z, x] \subseteq A$, *for any* $x \in A$, *is called starlikeness kernel of the set* A. *We denote the starlikeness kernel of the set* A *by* $\ker(A)$.

From Definitions 1, 4, 5 it follows directly:

Proposition 6. *(i) The set* $A \subseteq \mathbb{R}^n$ *is a starlike set if and only if* $\ker(A) \neq \emptyset$.
(ii) For any set $A \subseteq \mathbb{R}^n$, $\ker(A) \subseteq A$.
(iii) For any convex set $A \subseteq \mathbb{R}^n$, $\ker(A) = A$.
(iv) The starlikeness kernel of the set A, $\ker(A)$ *is a convex set.*

Definition 6. *Let* $X \subseteq \mathbb{R}^n$ *be a convex set. A function* $f : X \to \mathbb{R}$ *is called* convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for any $x, y \in X$ *and any* $\lambda \in (0, 1)$. *Let* $cx(X)$ *be the set of all real convex functions defined on the set* X.

As a generalization of the convex functions we consider the class of the starlike convex functions.

Definition 7. *Let* $X \subseteq \mathbb{R}^n$ *be a starlike set. A function* $f : X \to \mathbb{R}$ *is called* starlike convex *if there exists a point* $x^* \in X$ *such that*

$$f(\lambda x^* + (1-\lambda)y) \leq \lambda f(x^*) + (1-\lambda) f(y) \tag{1}$$

for any $y \in X$ *and any* $\lambda \in (0, 1)$ *and*

$$X_r = \{x \in X | f(x) \leq r\} \tag{2}$$

is a starlike set or an empty set for any $r \in \mathbb{R}$. *Let* $scx(X)$ *be the set of all real starlike convex functions defined on the set* X.

Definition 8. *Let $X \subseteq \mathbb{R}^n$ be a convex set. A function $f : X \to \mathbb{R}$ is called* quasi-convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$$

for any $x, y \in X$ and any $\lambda \in (0,1)$. Let $qcx(X)$ be the set of all real quasi-convex functions defined on the set X.

Theorem 7. *([6, 7]) Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is a convex and nonempty set. The function f is quasi-convex if and only if*

$$X_r = \{x \in X | f(x) \leq r\}$$

is convex for any $r \in \mathbb{R}$.

As a generalization of the quasi-convex function class, we consider the family of the starlike quasi-convex functions.

Definition 9. *Let $X \subseteq \mathbb{R}^n$ be a starlike set. A function $f : X \to \mathbb{R}$ is called* starlike quasi-convex *if there exists a point $x^* \in X$ such that*

$$f(\lambda x^* + (1-\lambda)y) \leq \max\{f(x^*), f(y)\} \tag{3}$$

for any $y \in X$ and any $\lambda \in (0,1)$ and

$$X_r = \{x \in X | f(x) \leq r\}$$

is a starlike set or an empty set for any $r \in \mathbb{R}$. Let $sqcx(X)$ be the set of all real starlike quasi-convex functions defined on the set X.

In the case of convex functions and more general of quasi-convex functions (see, Theorem 7) any level set X_r is convex.

Remark 1. *Concerning the starlike convex functions we note that the condition (1) only do not assure that the level sets X_r defined by (2) are all starlike sets.*

For instance, the function $f : X \to \mathbb{R}$, where

$$X = \{(x_1, 0), (0, x_2) : x_1 \in [-1,1], x_2 \in [-1,1]\} \subseteq \mathbb{R}^2$$

and

$$f(x_1, x_2) = \begin{cases} (x_1 - 0.5)^2, \text{ if } x_1 \in [0,1], x_2 = 0 \\ (x_1 + 0.5)^2, \text{ if } x_1 \in [-1,0], x_2 = 0 \\ (x_2 - 0.5)^2, \text{ if } x_2 \in [0,1], x_1 = 0 \\ (x_2 + 0.5)^2, \text{ if } x_2 \in [-1,0], x_1 = 0 \end{cases}$$

satisfies the condition (1) but it do not verify (2), because the level sets X_r for all $r \in [0, 0.25)$ are not starlike sets. Therefore, the function f below is not a starlike convex function. But, for instance, the function $f_1 : X \to \mathbb{R}$, where X is the same as in the preceding example and

$$f_1(x, x_2) = \begin{cases} x_1^2, \text{ if } x_1 \in [-1,1], x_2 = 0 \\ 2x_2, \text{ if } x_2 \in [-1,1], x = 0 \end{cases},$$

is a starlike convex function, since it satisfies both condition (1) and (2). The same remark is also true for starlike quasi-convex functions.

The function f_1 is a restriction to X of the convex function $f_2 : \mathbb{R}^2 \to \mathbb{R}$, where $f_2(x_1, x_2) = x_1^2 + 2x_2$.

Remark 2. *But not any restriction to a starlike set of a convex function is starlike convex.*

For instance, the function $f_3 : X \to \mathbb{R}$, where X is the same as in the preceding example and

$$f_3(x_1, x_2) = \begin{cases} (x_1 - 1)^2, \text{ if } x_1 \in [-1,1], x_2 = 0 \\ 2x_2, \text{ if } x_2 \in [-1,1], x_1 = 0 \end{cases}$$

is the restriction to X of the convex function $f_4 : \mathbb{R}^2 \to \mathbb{R}$, where $f_4(x_1, x_2) = (x_1 - 1)^2 + 2x_2$.

But the function f_3 is not a starlike convex function, because its level set

$$X_0 = \{(1,0)\} \bigcup \{(0, x_2) : x_2 \in [-1, 0]\}$$

is not a starlike set.

Remark 3. *There exists also a starlike convex function, which is not a restriction of a convex function.*

Let consider the function $f_5 : X^{'} \to \mathbb{R}$, where

$$X^{'} = X \bigcup \{(x_1, x_2) : x_1 = x_2, x_1 \in [-1,1], x_2 \in [-1,1]\}$$

and

$$f_5(x_1,x_2)=\begin{cases} x_1^2, \text{ if } x_1\in[-1,1], x_2=0\\ x_2^2, \text{ if } x_2\in[-1,1], x_1=0\\ 2(x_1+x_2), \text{ if } x_1=x_2\in[-1,1]\end{cases}.$$

The function f_5 is a starlike convex function but it is not a restriction to the set $X^{'}$ of a certain convex function, because

$$f_5(\tfrac{1}{2},\tfrac{1}{2})=2>\tfrac{1}{2}f_5(1,0)+\tfrac{1}{2}f_5(0,1)=1,$$

while the point $(\frac{1}{2},\frac{1}{2})=\frac{1}{2}(1,0)+\frac{1}{2}(0,1)$ is a convex combination of the points $(1,0)$ and $(0,1)$.

Between the families of convex, starlike convex, quasi-convex and starlike quasi-convex functions there exist the following relationships.

Theorem 8. *If $X\subseteq\mathbb{R}^n$ is a convex nonempty set, then the following inclusions hold*
$cx(X)\subseteq scx(X)\subseteq sqcx(X)$,
$cx(X)\subseteq qcx(X)\subseteq sqcx(X)$.

Proof. Since a convex set is a starlike set, from Definitions 6 and 7 it follows obviously the inclusion $cx(X)\subseteq scx(X)$. From Definitions 8 and 9, it results obviously the inclusion $qcx(X)\subseteq sqcx(X)$. The inclusion $cx(X)\subseteq qcx(X)$ is also well known, and follows immediately from Definitions 6 and 8. It remain to show only the inclusion $scx(X)\subseteq sqcx(X)$. But this inclusion holds, because the inequality (1) implies (3). ∎

We mention that Tigan [13], [14] was employed the class of starlike quasi-convex functions in order to prove some stability properties for optimization problem with respect to constraint perturbations.

Theorem 9. *If the infimum of a starlike quasi-convex function f defined on a starlike set $X\subseteq\mathbb{R}^n$ is finite and the minimum point set is non-empty, then the minimum point set of f is a starlike set.*

Proof. Let denote

$$r=\inf\{f(x):x\in X\}.$$

By theorem hypothesis $r \in \mathbb{R}$. Then since f is a starlike quasi-convex function it follows that the level set

$$X_r = \{x \in X | f(x) \leq r\}$$

is a starlike set. But since r is the infimum of the function f the minimum point set of f is the level set X_r. Therefore, the minimum point set of f is a starlike set. ∎

3. **Properties concerning starlike sets**

In this section, we will present the concepts of starlike hull and marginal points of a set which extend the notion of convex hull and extremal points of a set and we will give a generalization of the theorem 2, implying these concepts.

Definition 10. *(*[8, 9, 10]*) Let $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$ and $x^0 \in \mathbb{R}^n$. The intersection of all starlike sets in $\mathbb{R}^n$ with the starlikeness center x^0, that includes the set A is called the starlike hull with the starlikeness center x^0 of the set A. This set is denoted by $st(x^0, A)$.*

We can easily show that

$$st\left(x^0, A\right) = \bigcup_{x \in A} [x^0, x]. \tag{4}$$

Definition 11. *Let $K \neq \emptyset$, $K \subseteq \mathbb{R}^n$ and $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$. The set*

$$st(K, A) = \bigcup_{y \in K} st(y, A) \tag{5}$$

is called the starlike hull of the set A with respect to the starlikeness set K.

From Definition 11, it follows immediately the following theorem.

Theorem 10. *Let $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$. If $K^{'}$ is a nonempty set and $K^{'} \subseteq K^{''} \subseteq \mathbb{R}^n$, then $st(K^{'}, A) \subseteq st(K^{''}, A)$.*

Theorem 11. *If $B \subseteq A \subseteq \mathbb{R}^n$ and A is a starlike set having the starlikeness center x^0, then $st(x^0, B) \subseteq A$.*

Proof. Let $x \in B$. Since $x \in A$ and the set A is starlike, then $[x^0, x] \subseteq A$. Hence x is an arbitrary point in B, by (4), follows that $st(x^0, B) \subseteq A$. ∎

Theorem 12. *If $B \subseteq A \subseteq \mathbb{R}^n$, A is a starlike set and K is a nonempty set such that $K \subseteq \ker(A)$, then $st(K, B) \subseteq A$.*

Proof. By Theorem 11, $st(y, B) \subseteq A$ for any $y \in K$. Therefore, $\bigcup_{y\in K} st(y, B) \subseteq A$, i.e. $st(K, B) \subseteq A$. ∎

Definition 12. *Let A be a subset of $\mathbb{R}^n$ and $x^0 \in A$. A point $x^{'} \in A$ is called a marginal point of A with respect to x^0 if there are no $x^{''} \in A$, $x^{''} \neq x^0$ and $\lambda \in (0, 1]$, so that $x^{'} = \lambda x^0 + (1-\lambda)\, x^{''}$. We denote the set of all marginal points of A with respect to x^0 by $mg\left(x^0, A\right)$.*

Definition 13. *Let A be a subset of $\mathbb{R}^n$ and $K \subseteq A$. A point $x^{'} \in A$ is called a marginal point of A with respect to K if there are no two distinct points $y \in K, x^{''} \in A$ and $\lambda \in (0, 1]$, such that $x^{'} = \lambda y + (1-\lambda)\, x^{''}$. We denote the set of all marginal points of A with respect to K by $mg\,(K, A)$.*

From Definitions 12 and 13, it follows immediately the next property.

Theorem 13. *Let A be a subset of $\mathbb{R}^n$, $K^{'} \subseteq K^{''} \subseteq A$ and $K \subseteq A$. Then*
(i) $mg\,(K, A) = \bigcap_{y\in K} mg\,(y, A)\,,$
(ii) $mg\,(A, A) = ext(A),$
(iii) $mg\left(K^{''}, A\right) \subseteq mg(K^{'}, A).$

Proof. The assertion (i) obviously results from Definitions 12 and 13. The assertion (ii) follows from definitions 13 and 3. The point (iii) of the theorem is a direct consequence of the point (i). ∎

Definition 14. *Let $A \neq \emptyset$, $A \subseteq \mathbb{R}^n$ a starlike set. We define the marginal set of A as $mg(A) = mg(\ker(A), A)$.*

The following theorem represents a generalization to the starlike sets of the Minkowski's theorem (see, Theorem 2).

Theorem 14. *([8, 9, 10]) Any starlike and compact set A in $\mathbb{R}^n$, having the starlikeness centre x^0, is the starlike hull of the centre x^0 of $mg\left(x^0, A\right)$, i.e. $A = st(x^0, mg(x^0, A))$.*

Proof. From Definition 12 it follows that $mg\left(x^0, A\right) \subseteq A$. Since A is a starlike set, by Theorem 11, it results that

$$st\left(x^0, mg\left(x^0, A\right)\right) \subseteq A \tag{6}$$

Let $x \in A$. If $x \in mg(x^0, A)$, then obviously $x \in st\left(x^0, mg\left(x^0, A\right)\right)$. Let suppose that $x \notin mg\left(x^0, A\right)$. As, by hypothesis A is compact, there exists $x^{'} \in mg\left(x^0, A\right)$, so that $x = \lambda x^0 + (1-\lambda)\, x^{'}$ for a certain $\lambda \in (0, 1]$. Hence, in this case, $x \in st\left(x^0, mg\left(x^0, A\right)\right)$. Therefore, we have

$$A \subseteq st\left(x^0, mg\left(x^0, A\right)\right). \tag{7}$$

From (6) and (7), it follows that $A = st\left(x^0, mg\left(x^0, A\right)\right)$. ∎

Theorem 15. *Any starlike and compact set A in $\mathbb{R}^n$ is the starlike hull of the marginal point set $mg\,(A)$ with respect to* $\ker(A)$, *i.e.* $A = st(\ker(A), mg\,(A))$.

Proof. By theorem 13, it follows that $mg\,(A) = \bigcap_{y \in \ker(A)} mg\,(y, A)$. Therefore, $mg\,(A) \subseteq mg\,(y, A)$, for any $y \in \ker(A)$. Then, by Theorem 12, we have that $st(\ker(A), mg(A)) \subseteq st(\ker(A), mg(y, A))$ for any $y \in \ker(A)$. But, by Theorem 14 and (5), it results that $st(\ker(A), mg(y, A)) = A$. Therefore, we obtain

$$st(\ker(A), mg(A)) \subseteq A. \tag{8}$$

Let $x \in A$ be an arbitrary element of the set A. If $x \in \ker(A)$ or $x \in mg\,(A)$, then we evidently have $x \in st(\ker(A), mg(A))$. Let suppose that $x \notin \ker(A) \cup mg(A)$. Then, there exists $y \in \ker(A)$ and $x^{'} \in mg\,(A)$ such that $x = \lambda y + (1-\lambda)\, x^{'}$ for a certain $\lambda \in (0, 1)$. Therefore, $x \in st(\ker(A), mg(A))$, which implies the inclusion

$$A \subseteq st(\ker(A), mg(A)). \tag{9}$$

Otherwise, in virtue of Definitions 13 and 14, $x \in mg\,(A)$, which contradicts the above assumption. But, from (8) and (9) it results the theorem conclusion. ∎

Theorem 16. *If f is a continuous starlike quasi-convex function defined on a starlike compact set $X \subseteq \mathbb{R}^n$, then f has at least a maximum point in the set* $\ker(A) \cup mg(A)$.

Proof. Since f is a continuous function on a compact set X, f has at least a maximum point $x^* \in X$. If x^* is a marginal point of X then the thorem is true. Suppose that $x^* \notin mg(A)$. Then, there exists $y \in \ker(A)$ and $x^{'} \in mg(A)$ such that $x^* = \lambda y + (1-\lambda)x^{'}$ for a certain $\lambda \in (0,1)$.

Since f is starlike quasi-convex it follows that

$$f(x^*) \leq \max\{f\left(x^{'}\right), f(y)\}.$$

On the other hand, since x^* is a maximum point of f over the set X, we have

$$\max\{f\left(x^{'}\right), f(y) \leq f(x^*),$$

from where it follows that $\max\{f\left(x^{'}\right), f(y)\} = f(x^*)$. Therefore, at least one of the points $x^{'}$ or y is a maximum point, which implies the theorem conclusion. ∎

4. Application to the starlike quasi-convex programming problem with disjunctive constraints

Let $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{2, ..., m\}$ be starlike quasi-convex functions and let $\Omega = \{b_1, b_2, ..., b_s\} \subseteq \mathbb{R}^m$ be a finite set of vectors in $\mathbb{R}^m$.

By $g = (g_1, g_2, ..., g_m)^T$ we denote the vector of constraint functions for the starlike quasi-convex programming problem with disjunctive constraints (**QS**), and by b we denote $b = \inf \Omega$, where infimum is considered with respect to the usual order relation in $\mathbb{R}^n$.

QS. Find

$$\min f(x_1, x_2, ..., x_n)$$

submit to

$$(g(x_1, ..., x_n) \leq b_1) \vee (g(x_1, ..., x_n) \leq b_2) \vee ... \vee (g(x_1, ..., x_n) \leq b_s),$$
$$x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0.$$

Let S be the feasible set for problem **QS** and S_k and $S^{'}$ the sets:

$$S_k = \{x \in \mathbb{R}^n_+ | g(x) \leq b_k\},\ k = 1, 2, ..., s$$
$$S^{‘} = \{x \in \mathbb{R}^n_+ | g(x) \leq b\}.$$

Then it results that the following equalities hold

$$S = \cup_{k=1}^{s} S_k \tag{10}$$

$$S^{'} = \cap_{k=1}^{s} S_k \tag{11}$$

where S_k, by Theorem 7 and $S^{'}$, by Proposition 1 are both starlike sets, while the set S is generally not convex.

The set S has a property given by following theorem:

Theorem 17. *If the following two condition hold: (i) $S^{'} \neq \emptyset$,(ii) g_i, $i \in \{1, 2, ..., m\}$ are starlike quasi-convex functions, having all a common starlikeness point, then the feasible set S of problem **QS** is a starlike set.*

Proof. Choose an arbitrary point $x^0 \in S^{'}$ and let $x \in S$. By (10) it results that there exists $k \in \{1, 2, ..., s\}$ so that $x \in S_k$. Since S_k is a starlike set and, by (11), $x^0 \in S_k$, we have $[x^0, x] \subseteq S_k \subseteq S$. Therefore, the feasible set S is starlike. ∎

We note that the set S' represents a starlikeness kernel for the set S, as resulting from Theorem 17 and Definition 5.

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