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ON SOME CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

AMELIA ANCA HOLHOŞ

Abstract. In Holhoş [1] we defined and studied new classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ of univalent functions with negative coefficients for $0 \le \alpha < 1$, $0 < \beta \le 1$, $-1 \le A < B \le 1$, $0 < B \le 1$ and

$$\frac{B}{B-A} < \gamma \le \begin{cases} \dfrac{B}{(B-A)\alpha}, & \alpha \neq 0 \\ 1, & \alpha = 0 \end{cases}.$$

In this paper we study the classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ for $0 < \gamma \le \dfrac{B}{B-A}$.

1. Introduction

Let $\mathbf{U}$ denote the open unit disc: $\mathbf{U} = \{z\ ;\ z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathbf{S}$ denote the class of functions of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

which are analytic and univalent in $\mathbf{U}$.

For $f \in \mathbf{S}$ we define the differential operator $\mathbf{D}^n$ (Sălăgean [5])

$$\begin{aligned} \mathbf{D}^0 f(z) &= f(z) \\ \mathbf{D}^1 f(z) &= \mathbf{D}f(z) = zf'(z) \end{aligned}$$

and

$$\mathbf{D}^n f(z) = \mathbf{D}(\mathbf{D}^{n-1} f(z)) \quad ; \quad n \in \mathbb{N}^* = \{1,2,3,...\}.$$

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We note that if

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

then

$$\mathbf{D}^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j \ ; \ z \in \mathbf{U}.$$

Let T denote the subclass of $\mathbf{S}$ which can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \ ; \quad a_k \geq 0, \ \forall k \geq 2. \tag{1}$$

We say that a function $f \in T$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ $\quad 0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $0 < \gamma \leq \dfrac{B}{B-A}$ if

$$\left| \frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1}{(B-A)\gamma \left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - \alpha \right] - B \left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)} - 1 \right]} \right| < \beta \qquad z \in \mathbf{U}$$

where

$$F_{n,\lambda}(z) = (1-\lambda) D^n f(z) + \lambda D^{n+1} f(z) \ \ ; \ \lambda \geq 0 \ \ ; \ f \in T$$

Remark 1. *For $n = 0$, $\lambda = 0$, $A = -1$, $B = 1$, $\beta = 1$, the class* $T_{0,0}(-1, 1, \alpha, 1, \gamma) = S_0^*(\alpha, \gamma)$ *was studied by S.Owa* [2] *and for $n = 0$, $\lambda = 0$, $A = -1$, $B = 1$, the class* $T_{0,0}(-1, 1, \alpha, \beta, \gamma) = S_0^*(\alpha, \beta, \gamma)$ *was studied by S.Owa in* [3] *and* [4].

2. Characterization theorem

Theorem 2. *Let $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $0 < \gamma \leq \dfrac{B}{B-A}$. Then a function $f(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$; $a_k \geq 0, \forall k \geq 2$ is in the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty} a_k k^n \left[1 + \lambda(k-1)\right] \left[(k-1) + \beta B(k+1) - \beta\gamma(B-A)(k+\alpha)\right] \leq \\ \leq \ \ &\beta\gamma(B-A)(1-\alpha) \end{aligned} \tag{2}$$

and the result is sharp.

If we denote

$$\begin{aligned} &\mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) = \\ = \quad & k^n\left[1+\lambda\left(k-1\right)\right]\left[\left(k-1\right)+\beta B\left(k+1\right)-\beta\gamma\left(B-A\right)\left(k+\alpha\right)\right] \end{aligned} \tag{3}$$

then (2) becomes

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n\left(k, A, B, \alpha, \beta, \gamma, \lambda\right) \leq \beta\gamma(B-A)(1-\alpha).$$

Proof. Assume that (2) holds and let $|z| = 1$. Then we have

$$\begin{gathered} \left|zF'_{n,\lambda}\left(z\right)-F_{n,\lambda}\left(z\right)\right|- \\ -\beta\left|\left(B-A\right)\gamma\left[zF'_{n,\lambda}\left(z\right)-\alpha F_{n,\lambda}\left(z\right)\right]-B\left[zF'_{n,\lambda}\left(z\right)-F_{n,\lambda}\left(z\right)\right]\right| = \\ = \left|zF'_{n,\lambda}\left(z\right)-F_{n,\lambda}\left(z\right)\right|- \\ -\beta\left|\left[\left(B-A\right)\gamma-B\right]zF'_{n,\lambda}\left(z\right)+\left[B-(B-A)\gamma\alpha\right]F_{n,\lambda}\left(z\right)\right| = \\ = \left|\sum_{k=2}^{\infty}a_k k^n\left(1-k\right)\left[1+(k-1)\lambda\right]z^k\right|- \\ -\beta\left|\left[\left(B-A\right)\gamma-B\right]z-\left[\left(B-A\right)\gamma-B\right]\sum_{k=2}^{\infty}a_k k^{n+1}\left[1+(k-1)\lambda\right]z^k+\right. \\ \left.+\left[B-(B-A)\gamma\alpha\right]z-\left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]z^k\right| = \\ = \left|\sum_{k=2}^{\infty}a_k k^n\left(1-k\right)\left[1+(k-1)\lambda\right]z^k\right|- \\ -\beta\left|\left(B-A\right)\gamma\left(1-\alpha\right)z-\left[\left(B-A\right)\gamma-B\right]\sum_{k=2}^{\infty}a_k k^{n+1}\left[1+(k-1)\lambda\right]z^k-\right. \\ \left.-\left[B-(B-A)\gamma\alpha\right]\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]z^k\right| \leq \\ \leq \sum_{k=2}^{\infty}a_k k^n\left(k-1\right)\left[1+(k-1)\lambda\right]|z|^k-\beta\left(B-A\right)\gamma\left(1-\alpha\right)|z|+ \\ +\beta\left|\left(B-A\right)\gamma-B\right|\sum_{k=2}^{\infty}a_k k^{n+1}\left[1+(k-1)\lambda\right]|z|^k+ \end{gathered}$$

$$+\beta\left|B-(B-A)\gamma\alpha\right|\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]|z|^k\leq$$

$$\leq\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]\left\{(k-1)+\beta\left[B-(B-A)\gamma\right]k+\beta\left[B-(B-A)\gamma\alpha\right]\right\}-$$

$$-\beta\gamma(B-A)(1-\alpha)\leq 0$$

Consequently, by the maximum modulus theorem, the functions $f(z)$ is in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Conversely, assume that

$$\left|\frac{\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-1}{(B-A)\gamma\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-\alpha\right]-B\left[\dfrac{zF'_{n,\lambda}(z)}{F_{n,\lambda}(z)}-1\right]}\right|<\beta\Leftrightarrow$$

$$\left|\sum_{k=2}^{\infty}a_k k^n\left(1-k\right)\left[1+(k-1)\lambda\right]z^k\right|$$

$$\leq\ \beta\left|(B-A)\gamma\left(1-\alpha\right)z-\left[(B-A)\gamma-B\right]\sum_{k=2}^{\infty}a_k k^{n+1}\left[1+(k-1)\lambda\right]z^k-\right.$$

$$\left.-\left[B-(B-A\gamma\alpha)\right]\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]z^k\right|$$

Since $|\mathrm{Re}\,(z)|\leq|z|$ for any z, we have

$$\mathrm{Re}\left\{\frac{\sum\limits_{k=2}^{\infty}a_k k^n\left(k-1\right)\left[1+(k-1)\lambda\right]z^k}{(B-A)\gamma\left(1-\alpha\right)z-\sum\limits_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]\left[B(k+1)-(B-A)\gamma(k+\alpha)\right]z^k}\right\}<\beta$$

Letting $z\to 1$ through real values, upon clearing the denominator in the last inequality we obtain

$$\sum_{k=2}^{\infty}a_k k^n\left(k-1\right)\left[1+(k-1)\lambda\right]\leq$$

$$\leq\beta\gamma(B-A)(1-\alpha)-\sum_{k=2}^{\infty}a_k k^n\left[1+(k-1)\lambda\right]\beta\left[B(k+1)-(B-A)\gamma(k+\alpha)\right]$$

and this inequality gives the required condition.

Each function

$$f_k(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)}{k^n\left[1+\lambda\left(k-1\right)\right]\left[k-1+\beta B\left(k+1\right)-\beta\gamma\left(B-A\right)\left(k+\alpha\right)\right]} z^k$$

is an extremal function for the theorem. □

Remark 3. *For* $n=0,\ \lambda=0,\ A=-1,\ B=1,\ \beta=1$ *the result of Theorem 2 was obtained by Owa* [2] *and for* $n=0,\ \lambda=0,\ A=-1,\ B=1,$ *the result of Theorem 2 was obtain by Owa* [3] .

3. **Closure theorems**

Let the functions f_j be of the form:

$$f_j\left(z\right) = z - \sum_{k=2}^{\infty} a_{kj} z^k \quad ;\ z \in \mathbf{U} \quad ;\ a_{kj} \geq 0,\ \forall k \geq 2,\quad j=1,2,...,m. \tag{4}$$

We shall prove the following results for the closure of functions in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Theorem 4. *Let the functions* $f_j\left(z\right)$ *defined by* (4) *be in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. *Then the function* $g\left(z\right)$, *defined by*

$$g\left(z\right) = z - \sum_{k=2}^{\infty} b_k z^k \quad ;\ b_k \geq 0,\ \textit{with}\ b_k = \frac{1}{m}\sum_{j=1}^{m} a_{kj}$$

also belongs to the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Proof. As $f_j\left(z\right) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ it follows from Theorem 2 that

$$\sum_{k=2}^{\infty} a_{kj}\mathcal{D}_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right) \leq \beta\gamma(B-A)(1-\alpha) \quad ; \quad j=1,2,...,m.$$

Therefore

$$\begin{aligned}\sum_{k=2}^{\infty} b_k\mathcal{D}_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right) &= \sum_{k=2}^{\infty}\mathcal{D}_n\left(k,A,B,\alpha,\beta,\gamma,\lambda\right)\frac{1}{m}\sum_{j=1}^{m} a_{kj}\\ &\leq \beta\gamma(B-A)(1-\alpha)\end{aligned}$$

hence, by Theorem 2,

$$g(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma).$$

□

Theorem 5. *Let $f_j(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. Then the function $h(z)$, defined by,*

$$h(z) = \sum_{j=1}^{m} d_j f_j(z)\,; \text{ where } \sum_{j=1}^{m} d_j = 1,\ d_j \geq 0, \forall j = 1,2,...,m$$

is also in the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Proof. By using the definition of $h(z)$, we have

$$\begin{aligned} h(z) &= \sum_{j=1}^{m} d_j \left[z - \sum_{k=2}^{\infty} a_{kj} z^k \right] = z \sum_{j=1}^{m} d_j - \sum_{k=2}^{\infty}\sum_{j=1}^{m} d_j a_{kj} z^k = \\ &= z - \sum_{k=2}^{\infty} \left(\sum_{j=1}^{m} d_j a_{kj} z^k \right). \end{aligned}$$

Therefore,

$$\sum_{k=2}^{\infty} \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \left(\sum_{j=1}^{m} d_j a_{kj} \right) =$$

$$= \sum_{k=2}^{\infty} \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)\, a_{k1} d_1 + \sum_{k=2}^{\infty} \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)\, a_{k2} d_2 + ...$$

$$... + \sum_{k=2}^{\infty} \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)\, a_{km} d_m \leq$$

$$\leq d_1 \beta\gamma(B-A)(1-\alpha) + d_2 \beta\gamma(B-A)(1-\alpha) + ...$$

$$... + d_m \beta\gamma(B-A)(1-\alpha) =$$

$$= \beta\gamma(B-A)(1-\alpha) \sum_{j=1}^{m} d_j = \beta\gamma(B-A)(1-\alpha)$$

which implies that $h(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

Theorem 6. *Let the functions* $f_1(z) = z - \sum_{k=2}^{\infty} a_{k1} z^k$, $a_{k1} \geq 0, \forall k \geq 2$, *and* $f_2(z) = z - \sum_{k=2}^{\infty} a_{k2} z^k$, $a_{k2} \geq 0, \forall k \geq 2$ *be in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, *respectively* $T_{n+1,\lambda}(A,B,\alpha,\beta,\gamma)$. *Then the function* $p(z)$ *defined by*

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty}(a_{k1} + a_{k2}) z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma).$$

Proof. Let $f_1(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ and $f_2(z) \in T_{n+1,\lambda}(A,B,\alpha,\beta,\gamma)$; by using Theorem 2 we get, respectively,

$$\sum_{k=2}^{\infty} a_{k1}\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \beta\gamma(B-A)(1-\alpha)$$

and

$$\sum_{k=2}^{\infty} a_{k2}\mathcal{D}_{n+1}(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \beta\gamma(B-A)(1-\alpha)$$

We have (see (3))

$$\begin{aligned} 2\sum_{k=2}^{\infty} a_{k2}\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) &\leq \sum_{k=2}^{\infty} a_{k2}\mathcal{D}_{n+1}(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \\ &\leq \beta\gamma(B-A)(1-\alpha). \end{aligned}$$

Then

$$\frac{2}{3}\sum_{k=2}^{\infty} a_{k1}\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \frac{2}{3}\beta\gamma(B-A)(1-\alpha)$$

and

$$\frac{2}{3}\sum_{k=2}^{\infty} a_{k2}\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \frac{1}{3}\beta\gamma(B-A)(1-\alpha)$$

imply

$$\frac{2}{3}\sum_{k=2}^{\infty}(a_{k1} + a_{k2})\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \leq \beta\gamma(B-A)(1-\alpha),$$

and from this we deduce that

$$p(z) = z - \frac{2}{3}\sum_{k=2}^{\infty}(a_{k1} + a_{k2}) z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma).$$

□

4. **Integral Operators**

Theorem 7. *Let the function* $f(z)$ *defined by* (1), *be in the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, *and let* c *be a real number such that* $c > -1$. *Then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt \tag{5}$$

also belongs to the class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$.

Proof. By using the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \ \ where \ \ b_k = \frac{c+1}{c+k} a_k$$

$$f \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \Rightarrow \sum_{k=2}^{\infty} a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \le \beta\gamma(B-A)(1-\alpha)$$

$$\begin{aligned}\sum_{k=2}^{\infty} b_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) &= \sum_{k=2}^{\infty} \frac{c+1}{c+k} a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) < \\ &< \sum_{k=2}^{\infty} a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda) \\ &\le \beta\gamma(B-A)(1-\alpha)\end{aligned}$$

$\Rightarrow F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

Theorem 8. *Let* c *be a real number such that* $c > -1$. *If* $F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *then the function* $f(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt$$

is univalent in $|z| < R$, *where*

$$R = \inf_k \left[\frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)(c+k)k}\right]^{\frac{1}{k-1}}, \ k \ge 2 \tag{6}$$

The result is sharp for

$$f(z) = z - \frac{\beta\gamma(B-A)(1-\alpha)(c+k)z^k}{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}, \qquad k \ge 2 \tag{7}$$

Proof. Let $F(z) = z - \sum\limits_{k=2}^{\infty} a_k z^k$; it follows from (5) that

$$f(z) = \frac{z^{1-c}\left[z^c F(z)\right]'}{c+1} = z - \sum_{k=2}^{\infty} \frac{c+k}{c+1} a_k z^k$$

$$F(z) \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma) \Rightarrow \sum_{k=2}^{\infty} \frac{a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \leq 1.$$

If

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)}$$

or if

$$|z| < \left[\frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)(c+1)}{\beta\gamma(B-A)(1-\alpha)k(c+k)}\right]^{\frac{1}{k-1}}$$

then

$$\begin{aligned} |f'(z) - 1| &= \left| -\sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k z^{k-1} \right| \leq \sum_{k=2}^{\infty} k \frac{c+k}{c+1} a_k |z|^{k-1} < \\ &< \sum_{k=2}^{\infty} \frac{a_k \mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \leq 1 \end{aligned}$$

But from $|f'(z) - 1| < 1$, $|z| < R$, we deduce that f is univalent in the disc $|z| < R$.

The result is sharp and the extremal function is given by (7). □

Theorem 9. *Let $c \in \mathbb{R}$, $c > -1$. If*

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k \in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$$

then the function $f(z)$ given by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt$$

is starlike of order p $(0 \leq p < 1)$ in $|z| < R^*(p,A,B,\alpha,\beta,\gamma)$ where

$$R^* = \inf_k \left[\frac{(1-p)(c+1)\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad ; \quad k \geq 2.$$

The result is sharp.

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-1\right|<(1-p)$, in $|z|<R^*$.

Now

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{-\sum\limits_{k=2}^{\infty}(k-1)\frac{c+k}{c+1}a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{c+k}{c+1}a_k z^{k-1}}\right| \le \frac{\sum\limits_{k=2}^{\infty}(k-1)\frac{c+k}{c+1}a_k |z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}\frac{c+k}{c+1}a_k |z|^{k-1}} < 1-p$$

provided

$$\sum_{k=2}^{\infty}\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)a_k |z|^{k-1} < 1$$

By using

$$\sum_{k=2}^{\infty}\frac{a_k\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \le 1$$

the inequality

$$\sum_{k=2}^{\infty}\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)a_k |z|^{k-1} < 1$$

holds if

$$\left(\frac{k-p}{1-p}\right)\left(\frac{c+k}{c+1}\right)|z|^{k-1} < \frac{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{\beta\gamma(B-A)(1-\alpha)} \quad ; \quad k\ge 2$$

or if

$$|z| < \left[\frac{(1-p)(c+1)\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)}{(k-p)(c+k)\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad ; \quad k\ge 2.$$

Hence, $f(z)\in S^*(p)$ in $|z|<R^*$. The sharpness follows if we take the function $F(z)$, given by

$$F(z) = z - \frac{(B-A)\gamma\beta(1-\alpha)z^k}{\mathcal{D}_n(k,A,B,\alpha,\beta,\gamma,\lambda)},\ k\ge 2.$$

□

5. **The Hadamard products**

Let $f,g\in T$,

$$f(z) = z - \sum_{k=2}^{\infty}a_k z^k;\ a_k\ge 0,\ \forall k\ge 2 \tag{8}$$

and

$$g(z) = z - \sum_{k=2}^{\infty}b_k z^k;\ b_k\ge 0,\ \forall k\ge 2, \tag{9}$$

then we define the Hadamard product of f and g by

$$f * g(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k.$$

Theorem 10. *If the functions f and g defined by* (8) *and* (9) *belong to the same class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$, then the Hadamard product $f * g$ belongs to the class $T_{n,\lambda}(A, B, \alpha, \beta^2, \gamma)$.*

Proof. Since $f(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ by using Theorem 2 we have

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

and

$$a_k \leq \frac{\beta\gamma(B - A)(1 - \alpha)}{2^n(1 + \lambda)[1 + 3\beta B - \beta\gamma(B - A)(2 + \alpha)]} \ ; \qquad \forall k \geq 2.$$

If $g(z) \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ then

$$\sum_{k=2}^{\infty} b_k \mathcal{D}_n (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

Since $0 < \beta^2 \leq \beta \leq 1$ we have

$$\sum_{k=2}^{\infty} b_k \mathcal{D}_n \left(k, A, B, \alpha, \beta^2, \gamma, \lambda\right) \leq \sum_{k=2}^{\infty} b_k \mathcal{D}_n (k, A, B, \alpha, \beta, \gamma, \lambda)$$

and then

$$\begin{aligned} \sum_{k=2}^{\infty} a_k b_k \mathcal{D}_n \left(k, A, B, \alpha, \beta^2, \gamma, \lambda\right) &\leq \sum_{k=2}^{\infty} a_k b_k \mathcal{D}_n (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \\ &\leq \frac{\beta^2\gamma^2(B - A)^2(1 - \alpha)^2}{2^n(1 + \lambda)[1 + 3\beta B - \beta\gamma(B - A)(2 + \alpha)]} \leq \\ &\leq \beta^2\gamma(B - A)(1 - \alpha) \end{aligned}$$

because

$$\frac{\beta^2\gamma^2(B - A)^2(1 - \alpha)^2}{2^n(1 + \lambda)[1 + 3\beta B - \beta\gamma(B - A)(2 + \alpha)]} \leq \beta^2\gamma(B - A)(1 - \alpha) \Leftrightarrow$$

$$\beta^2\gamma(B-A)(1-\alpha)\{\gamma(B - A)(1 - \alpha) - 2^n(1 + \lambda)[1 + 3\beta B - \beta\gamma(B - A)(2 + \alpha)]\} \leq 0$$

$$\Leftrightarrow \gamma(B - A)(1 - \alpha) - 2^n(1 + \lambda) - 2^n(1 + \lambda)\beta B - 2^n(1 + \lambda)2\beta B +$$

$$+2^n(1 + \lambda)\beta\gamma(B - A)2 + 2^n(1 + \lambda)\beta\gamma(B - A)\alpha \leq 0$$

$$\gamma\,(B-A)\,(1-\alpha)-2^n(1+\lambda)+2^n(1+\lambda)\beta\,[\gamma(B-A)\alpha-B]+$$

$$+2^n(1+\lambda)2\beta\,[\gamma(B-A)-B]\leq 0$$

According to Theorem 2 we obtain $f*g\in T_{n,\lambda}(A,B,\alpha,\beta^2,\gamma)$. □

Theorem 11. *Let $p>0$ and $\frac{p+2-\sqrt{p^2+4p}}{2}\leq\alpha\leq\frac{1}{1+p}$. If the functions f and g defined by* (8) *and* (9) *belong to the same class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, then the Hadamard product $f*g$ belongs to the class $T_{n,\lambda}(A,B,1-p\alpha,\beta^2,\gamma)$.*

Proof. By using

$$a_k\leq\frac{\beta\gamma(B-A)(1-\alpha)}{2^n(1+\lambda)\,[1+3\beta B-\beta\gamma(B-A)(2+\alpha)]}\;;\qquad\forall k\geq 2$$

and

$$\sum_{k=2}^{\infty}b_k\mathcal{D}_n\,(k,A,B,\alpha,\beta,\gamma,\lambda)\leq\beta\gamma(B-A)(1-\alpha)$$

we obtain

$$\begin{aligned}\sum_{k=2}^{\infty}a_kb_k\mathcal{D}_n\left(k,A,B,1-\alpha,\beta^2,\gamma,\lambda\right) &\leq \sum_{k=2}^{\infty}a_kb_k\mathcal{D}_n\left(k,A,B,\alpha,\beta^2,\gamma,\lambda\right)\leq\\ &\leq \sum_{k=2}^{\infty}a_kb_k\mathcal{D}_n\,(k,A,B,\alpha,\beta,\gamma,\lambda)\leq\\ &\leq \frac{\beta^2\gamma^2(B-A)^2(1-\alpha)^2}{2^n(1+\lambda)\,[1+3\beta B-\beta\gamma(B-A)(2+\alpha)]}\leq\\ &\leq \beta^2\gamma(B-A)(1-\alpha)^2\leq\beta^2\gamma(B-A)\alpha\end{aligned}$$

which implies that $f*g\in T_{n,\lambda}(A,B,1-\alpha,\beta^2,\gamma)$. □

Corollary 12. *Let $\frac{3-\sqrt{5}}{2}\leq\alpha\leq\frac{1}{2}$ and let the functions f and g defined by* (8) *and* (9) *be in the same class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. Then the Hadamard product $f*g$ belongs to the class $T_{n,\lambda}(A,B,1-\alpha,\beta^2,\gamma)$.*

Corollary 13. *Let $2-\sqrt{3}\leq\alpha\leq\frac{1}{3}$ and let the functions f and g defined by* (8) *and* (9) *be in the same class $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. Then the Hadamard product $f*g$ belongs to the class $T_{n,\lambda}(A,B,1-2\alpha,\beta^2,\gamma)$.*

Remark 14. *From definition of the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *it is easy to see that if* $0<\beta_1\leq\beta_2\leq 1$ *then* $T_{n,\lambda}(A,B,\alpha,\beta_1,\gamma)\subset T_{n,\lambda}(A,B,\alpha,\beta_2,\gamma)$.

Remark 15. *Since* $0<\beta^2\leq\beta\leq 1$ *we have* $T_{n,\lambda}(A,B,\alpha,\beta^2,\gamma)\subset T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ *and* $T_{n,\lambda}(A,B,1-p\alpha,\beta^2,\gamma)\subset T_{n,\lambda}(A,B,1-p\alpha,\beta,\gamma)$.

6. Inclusion properties of the classes $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$

Theorem 16. *Let* $0\leq\alpha_2\leq\alpha_1<1$; $0<\beta_1\leq\beta_2\leq 1$ *and* $0<\gamma_1\leq\gamma_2\leq\dfrac{B}{B-A}$. *Then we have* $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$.

Proof. Let $f\in T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)$. Then, by using Theorem 2, we have

$$\sum_{k=2}^{\infty}a_k\mathcal{D}_n\left(k,A,B,\alpha_1,\beta_1,\gamma_1,\lambda\right)\leq\beta_1\gamma_1(B-A)(1-\alpha_1). \tag{10}$$

From this we deduce that

$$\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]\left[B\left(k+1\right)-\gamma_1(B-A)(k+\alpha_1)\right]\leq\gamma_1(B-A)(1-\alpha_1),$$

$$\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]\left[-(k+\alpha_1)\right]\leq 1-\alpha_1$$

and

$$\sum_{k=2}^{\infty}a_kk^n\left[1+\lambda\left(k-1\right)\right]\leq 1.$$

Let $\alpha_1=\alpha_2+\delta$; $\beta_1=\beta_2-\varepsilon$; $\gamma_1=\gamma_2-\theta$ where $\delta,\varepsilon,\theta\geq 0$, then from (10) we have

$$\sum_{k=2}^{\infty}a_k\mathcal{D}_n\left(k,A,B,\alpha_2+\delta,\beta_2-\varepsilon,\gamma_2-\theta,\lambda\right)\leq\beta_1\gamma_1(B-A)(1-\alpha_1)$$

and because

$$\mathcal{D}_n\left(k,A,B,\alpha_2+\delta,\beta_2-\varepsilon,\gamma_2-\theta,\lambda\right)=\mathcal{D}_n\left(k,A,B,\alpha_2,\beta_2,\gamma_2,\lambda\right)-$$

$$-k^n\left[1+\lambda\left(k-1\right)\right]\varepsilon\left[B\left(k+1\right)-\gamma_1\left(B-A\right)\left(k+\alpha_1\right)\right]-$$

$$-k^n\left[1+\lambda\left(k-1\right)\right]\beta_2\theta\left(B-A\right)\left[-\left(k+\alpha_1\right)\right]-k^n\left[1+\lambda\left(k-1\right)\right]\gamma_2\beta_2\left(B-A\right)\delta$$

we have

$$\sum_{k=2}^{\infty}a_k\mathcal{D}_n\left(k,A,B,\alpha_2,\beta_2,\gamma_2,\lambda\right)\leq\beta_1\gamma_1(B-A)(1-\alpha_1)+$$

$$+\varepsilon\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[B\left(k+1\right)-\gamma_1\left(B-A\right)\left(k+\alpha_1\right)\right]+$$

$$+\beta_2\theta\left(B-A\right)\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\left[-\left(k+\alpha_1\right)\right]+$$

$$+\gamma_2\beta_2\left(B-A\right)\delta\sum_{k=2}^{\infty}a_k k^n\left[1+\lambda\left(k-1\right)\right]\leq$$

$$\leq\beta_1\gamma_1(B-A)(1-\alpha_1)+\varepsilon\gamma_1(B-A)(1-\alpha_1)+\beta_2\theta\left(B-A\right)\left(1-\alpha_1\right)+$$

$$+\beta_2\gamma_2\left(B-A\right)\delta=\beta_2\gamma_2(B-A)(1-\alpha_2)$$

According to Theorem 2 we obtain $f\in T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$ and $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$. □

Corollary 17. *Let* $0\leq\alpha_1\leq\alpha_2<1$. *Then we have* $T_{n,\lambda}(A,B,\alpha_1,\beta,\gamma)\supset T_{n,\lambda}(A,B,\alpha_2,\beta,\gamma)$.

Corollary 18. *Let* $0<\gamma_1\leq\gamma_2\leq\dfrac{B}{B-A}$. *Then*

$$T_{n,\lambda}(A,B,\alpha,\beta,\gamma_1)\subset T_{n,\lambda}(A,B,\alpha,\beta,\gamma_2).$$

Theorem 19. *Let* $0\leq\alpha_2\leq\alpha_1<1$; $0<\beta_1\leq\beta_2\leq1$ *and* $0<\gamma_1\leq\gamma_2\leq\dfrac{B}{B-A}$. *If the functions* f *defined by* (8) *and* g *defined by* (9) *be in the classes* $T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)$ *and* $T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)$, *respectively, then the Hadamard product* $f*g$ *belongs to the class* $T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$, *where* $\alpha=\min\left(\alpha_1,\alpha_2\right)$, $\beta=\max\left(\beta_1,\beta_2\right)$ *and* $\gamma=\max\left(\gamma_1,\gamma_2\right)$.

Proof. Since

$$\alpha=\min\left(\alpha_1,\alpha_2\right)\Rightarrow\alpha\leq\alpha_1\text{ and }\ \alpha\leq\alpha_2$$

$$\beta=\max\left(\beta_1,\beta_2\right)\Rightarrow\beta\geq\beta_1\text{ and }\beta\geq\beta_2$$

$$\gamma=\max\left(\gamma_1,\gamma_2\right)\Rightarrow\gamma\geq\gamma_1\text{ and }\gamma\geq\gamma_2$$

from Theorem 16 we have $f\in T_{n,\lambda}(A,B,\alpha_1,\beta_1,\gamma_1)\Rightarrow f\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$ and $g\in T_{n,\lambda}(A,B,\alpha_2,\beta_2,\gamma_2)\Rightarrow g\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. From Theorem 10 and Remark 15 we have $f*g\in T_{n,\lambda}(A,B,\alpha,\beta,\gamma)$. □

Theorem 20. *Let* $-1\leq A_2\leq A_1<B_1\leq B_2\leq1$, $0<B_1$. *Then we have* $T_{n,\lambda}(A_1,B_1,\alpha,\beta,\gamma)\subset T_{n,\lambda}(A_2,B_2,\alpha,\beta,\gamma)$.

Proof. Let $f \in T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma)$ then

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n (k, A_1, B_1, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B_1 - A_1)(1 - \alpha).$$

Since $A_1 \geq A_2$, $B_1 \leq B_2 \Rightarrow B_1 - A_1 \leq B_2 - A_2$ and because $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq \dfrac{B}{B - A}$ from (3) we deduce that $\mathcal{D}_n (k, A_1, B_1, \alpha, \beta, \gamma, \lambda) \geq \mathcal{D}_n (k, A_2, B_2, \alpha, \beta, \gamma, \lambda)$. We have

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n (k, A_2, B_2, \alpha, \beta, \gamma, \lambda) \leq \sum_{k=2}^{\infty} a_k \mathcal{D}_n (k, A_1, B_1, \alpha, \beta, \gamma, \lambda) \leq$$

$$\leq \beta\gamma(B_1 - A_1)(1 - \alpha) \leq \beta\gamma(B_2 - A_2)(1 - \alpha),$$

and according to Theorem 2 we obtain $f \in T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$ which imply that $T_{n,\lambda}(A_1, B_1, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A_2, B_2, \alpha, \beta, \gamma)$. □

Corollary 21. *Let* $-1 \leq A_2 \leq A_1 < B \leq 1$, $0 < B \leq 1$. *Then we have* $T_{n,\lambda}(A_1, B, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A_2, B, \alpha, \beta, \gamma)$.

Corollary 22. *Let* $-1 \leq A < B_1 \leq B_2 \leq 1$, $0 < B_1 \leq B_2 \leq 1$. *Then we have* $T_{n,\lambda}(A, B_1, \alpha, \beta, \gamma) \subset T_{n,\lambda}(A, B_2, \alpha, \beta, \gamma)$.

Theorem 23. $T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \supset T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$.

Proof. Since $f(z) \in T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$ by using Theorem 2 we have

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_{n+1} (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

because

$$k^n < k^{n+1};\ \forall k \geq 2 \text{ and } \forall n \geq 0$$

and

$$k^n \left[1 + \lambda (k - 1)\right] \left[(k - 1) + \beta B (k + 1) - \beta\gamma (B - A) (k + \alpha)\right] > 0$$

then

$$\mathcal{D}_n (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \mathcal{D}_{n+1} (k, A, B, \alpha, \beta, \gamma, \lambda) \ ; \quad \forall n \geq 0$$

and

$$\sum_{k=2}^{\infty} a_k \mathcal{D}_n (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \sum_{k=2}^{\infty} a_k \mathcal{D}_{n+1} (k, A, B, \alpha, \beta, \gamma, \lambda) \leq \beta\gamma(B - A)(1 - \alpha)$$

According to Theorem 2 we obtain $f \in T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \Rightarrow T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \supset T_{n+1,\lambda}(A, B, \alpha, \beta, \gamma)$. □

Remark 24. *From definition of the class $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ or from Theorem 2 it is easy to see that if $0 \leq \lambda_1 \leq \lambda_2$ then $T_{n,\lambda_2}(A, B, \alpha, \beta, \gamma) \subset T_{n,\lambda_1}(A, B, \alpha, \beta, \gamma)$.*

Remark 25. *From Theorem 23 we have*

$$T_{n,\lambda}(A, B, \alpha, \beta, \gamma) \subset T_{0,\lambda}(A, B, \alpha, \beta, \gamma),$$

from Remark 24 we have

$$T_{0,\lambda}(A, B, \alpha, \beta, \gamma) \subset T_{0,0}(A, B, \alpha, \beta, \gamma)$$

and from Theorem 16 and Theorem 20 we have

$$T_{0,0}(A, B, \alpha, \beta, \gamma) \subset T_{0,0}(-1, 1, 0, 1, \frac{1}{2})$$

and

$$f \in T_{0,0}(-1, 1, 0, 1, \frac{1}{2}) \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1$$

the class of starlike functions with negative coefficients. Because these functions are univalent, then all functions in the classes $T_{n,\lambda}(A, B, \alpha, \beta, \gamma)$ are univalent.

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Universitatea Ecologică Deva, Romania