STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 1, March 2004

A NEW CONVEXITY CRITERION

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Abstract. In this paper we have obtained a simple sufficient condition for the convexity of analytic functions defined in the unit disc $U = \{z \in C : |z| < 1\}$.

1. **Introduction**

Let A be the class of functions which are analytic in the unit disc $U = \{z \in \mathbb{C} \mid |z| < 1\}$ and has the form $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$.
The analytic function f is said to be in the class $\mathcal{P}$, if and only if $f(0) = 1$ and $Re\, f(z) > 0$, $\forall\ z \in U$. If f and g are analytic in the unit disc U,we say that f is subordinate to g in U if there exist a function Φ analytic in U, so that $\Phi(0) = 0$, $|\Phi(z)| < 1$, and $f(z) = g(\Phi(z))$ for all $z \in U$. The subordination shall be denoted by $f \prec g$. If g is univalent, then f is subordinated to g if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.
We say that the analytic function f is convex in U if it is univalent and $f(U)$ is a convex domain in $\mathbb{C}$. It is well known that a function f is convex if and only if $f'(0) \neq 0$ and $Re\left(1 + \frac{z f''(z)}{f'(z)}\right) > 0$, for all $z \in U$. Let K denote the subset of A consisting of convex functions. In order to show our main result,we need the following lemmas.

Lemma 1. *(Herglotz) [1]*
A function f belongs to $\mathcal{P}$ if and only if there is a measure μ on $[0, 2\pi]$ so that

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, \mathrm{d}\mu(t) \quad \textit{and} \quad \mu\left([0, 2\pi]\right) = 1 \ .$$

Lemma 2. *(H.S.Wilf)[4]*
If $Re\left(\frac{1}{2} + \sum_{n=1}^{\infty} b_n z^n\right) > 0, \forall\, z \in U$ *and* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *is a convex function, then* $\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z)$.

Received by the editors: 03.12.2002.

2. **Main result**

Theorem 1. If $f \in A$ and $Re\left(zf''(z) + \frac{z^2}{2}f'''(z)\right) > \frac{-1}{\pi + 4\ln 2}$ for $z \in U$ then $f \in K$.

Proof. If $f \in A$ then it has the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. The condition $Re\left(zf''(z) + \frac{z^2}{2}f'''(z)\right) >$
$> \frac{-1}{\pi+4\ln 2} = -c$ is equivalent with $1 + \frac{1}{c}\left(zf''(z) + \frac{z^2}{2}f'''(z)\right) \in \mathcal{P}$ and using the Lemma 1 we obtain the following representation:

$$1 + \frac{1}{2c}\sum_{n=2}^{\infty} n^2(n-1)a_n z^{n-1} = 1 + 2\sum_{n=2}^{\infty} z^{n-1}\int_0^{2\pi} e^{-it(n-1)}\ \mathrm{d}\mu(t)$$

From the last equality we deduce that:

$$a_n = \frac{4c}{n^2(n-1)}\int_0^{2\pi} e^{-it(n-1)}\ \mathrm{d}\mu(t)$$

and

$$f(z) = z + 4c\sum_{n=2}^{\infty} \frac{z^n}{n^2(n-1)}\int_0^{2\pi} e^{-it(n-1)}\ \mathrm{d}\mu(t)$$

After a simple calculation we get that

$$1 + \frac{zf''(z)}{f'(z)} = \frac{\dfrac{1}{4c} + \displaystyle\sum_{n=2}^{\infty} \frac{z^{n-1}}{n-1}\int_0^{2\pi} e^{-it(n-1)}\ \mathrm{d}\mu(t)}{\dfrac{1}{4c} + \displaystyle\sum_{n=2}^{\infty} \frac{z^{n-1}}{n(n-1)}\int_0^{2\pi} e^{-it(n-1)}\ \mathrm{d}\mu(t)}$$

We introduce the notations $h(z) = \frac{1}{4c} + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n-1}$ and $g(z) = \frac{1}{4c} + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n(n-1)}$. It is easy to observe that $h(z) = \frac{1}{4c} + \log\frac{1}{1-z}$ and h is a convex function.
Because $Re\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{z^n}{n+1}\right) > 0$, $\forall\, z \in U$ and h is convex using Lemma 2 it follows that $g(z) \prec h(z)$.
The convexity of h implies

$$\frac{1}{4c} + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n(n-1)}\int_0^{2\pi} e^{-it(n-1)}\ \mathrm{d}\mu(t) \in h(U), \tag{1}$$

$$\frac{1}{4c}+\sum_{n=2}^{\infty}\frac{z^{n-1}}{n-1}\int_0^{2\pi}e^{-it(n-1)}\,\mathrm{d}\mu(t)\in h(U) \tag{2}$$

for every $z\in U$ and every measure μ for which $\mu([0,2\pi])=1$.

If $0<c<\dfrac{1}{4}$ then $Re\ h(z)>0, z\in U$ and we can draw two tangent lines to the curve $\Gamma=\partial h(U)$. Let denote with α the measure of the angle between the two tangent lines which contains $h(U)$. From(1) and (2) follows that

$$|\arg\frac{\dfrac{1}{4c}+\displaystyle\sum_{n=2}^{\infty}\frac{z^{n-1}}{n-1}\int_0^{2\pi}e^{-it(n-1)}\,\mathrm{d}\mu(t)}{\dfrac{1}{4c}+\displaystyle\sum_{n=2}^{\infty}\frac{z^{n-1}}{n(n-1)}\int_0^{2\pi}e^{-it(n-1)}\,\mathrm{d}\mu(t)}|<\alpha$$

and so a sufficient condition for $Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0,\ z\in U$ is $\alpha=\dfrac{\pi}{2}$.

The curve Γ has the $h(e^{i\theta})=u(\theta)+iv(\theta)$, $\theta\in(0,2\pi)$ parametric representation.

The equality $\alpha=\dfrac{\pi}{2}$ is equivalent with the existence of $\theta_1,\theta_2\in[0,2\pi]$ with the properties

$$\frac{u(\theta_1)}{u'(\theta_1)}=\frac{v(\theta_1)}{v'(\theta_1)}\ ,\ \frac{u(\theta_2)}{u'(\theta_2)}=\frac{v(\theta_2)}{v'(\theta_2)}\ ,\ \frac{v'(\theta_1)}{u'(\theta_1)}\cdot\frac{v'(\theta_2)}{u'(\theta_2)}=-1$$

Because $h(U)$ is symmetric with respect to the real axis,we deduce that $\theta_2=2\pi-\theta_1$ and after calculation we get $c=\frac{1}{\pi+2\ln 2}$. ☐

Example. *Let λ be a real number so that $0<\lambda<\frac{e-1}{e(\pi+2\ln 2)}$ then the function* $f(z)=z\,+\lambda\displaystyle\int_0^z\!\!\int_0^t\frac{1}{u^2}\int_0^u\frac{s^2}{e^s-1}\,\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}t$ belongs to K.

Proof. After derivation we get that: $zf''(z)+\dfrac{z^2}{2}f'''(z)=\lambda\dfrac{z^2}{e^z-1},\ z\in U$. In [1] had been proved that $q(z)=\dfrac{e^z-1}{z}$ is a convex function in U which implies the inequality:

$$Re\ q(z)>\frac{e-1}{e}\ ,\ z\in U. \tag{2}$$

From (2) follows that $|q(z)|>\dfrac{e-1}{e}, z\in U$ or equivalently

$$\left|\frac{z}{e^z-1}\right|<\frac{e}{e-1}\ ,\ z\in U. \tag{3}$$

Using (3) it is easy to deduce that

$$Re\left(zf''(z)+\frac{z^2}{2}f'''(z)\right)=\lambda\ Re\frac{z^2}{e^z-1}>\lambda\frac{-e}{e-1}=\frac{-1}{\pi+2\ln 2},\ z\in U$$

which is the condition of Theorem 1. ☐

References

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