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ON A CLASS OF GENERALIZED GAUSS-CHRISTOFFEL QUADRATURE FORMULAE

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Abstract. We consider Gauss-Christoffel-Stancu quadrature rules, over the interval $[-1,1]$, using m Gaussian nodes and s preassigned multiples nodes, so that the node polynomial of these fixed nodes does not change sign in $(-1,1)$. The Gaussian nodes x_k of formula (2) are determined so that the degree of exactness of this quadrature formula to be the highest possible. These can be found either by means of the formula (10) or by determining the minimum of the function F of m variables (11). We give explicit formulae for the coefficients and for the remainders. Several illustrative examples are presented for certain preassigned multiple nodes.

1. In a memoir published by E. B. Christoffel in 1858 [1] has been considered a generalization of the classical Gauss quadrature formula, by introducing certain preassigned simple nodes situated outside the integration interval $(-1,1)$.

This formula has the following form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{j=1}^{n} B_j f(b_j) + R(f), \tag{1}$$

where b_j are preassigned nodes (the fixed nodes), not situated in the interval $(-1,1)$, f is an integrable function on this interval and $R(f)$ is the remainder of this quadrature formula. The free nodes x_k are selected so that formula (1) has the highest degree of exactness. We will call x_k the **fundamental** or the **Gaussian nodes**.

2. In 1957 D. D. Stancu [4] has introduced and investigated a quadrature formula using multiple fixed nodes a_i and simple Gaussian nodes x_k.

It has the form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f). \tag{2}$$

Let us denote by $u(x)$ the node polynomial of the free nodes x_k and by $\omega(x)$ the node polynomial of the fixed nodes, that is

$$u(x) = (x-x_1)(x-x_2)\dots(x-x_m), \tag{3}$$

$$\omega(x) = (x-a_1)^{r_1}(x-a_2)^{r_2}\dots(x-a_s)^{r_s}. \tag{4}$$

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We assume that r_i are natural numbers so that we have $\omega(x) \geqq 0$ on the integration interval $(-1, 1)$.

Given the fixed nodes a_i and their multiplicities r_i, the problem is then to determine the simple nodes x_k and the coefficients A_k and $C_{i,j}$ so that formula (2) has the highest degree of exactness.

In order to find the Gaussian nodes x_k we shall start from the Lagrange-Hermite interpolation formula using the simple nodes x_k, the multiple nodes a_i and other nondetermined simple nodes $t_1, t_2, \dots, t_m$, distinct from the other nodes. It has the form

$$f(x) = (H_{2m+p-1}f)(x) + (R_{2m+p-1}f)(x), \tag{5}$$

where we use as nodes the roots of the polynomial $P(x) = u(x)\omega(x)v(x)$, u and ω being defined at (3) and (4), while

$$v(x) = (x - t_1)(x - t_2)\dots(x - t_m), \quad p = r_1 + r_2 + \cdots + r_s.$$

The interpolation polynomial H_{2m+p-1} has the following expression (see [4]):

$$(H_{2m+p-1}f)(x) = \sum_{k=1}^{m} \frac{u_k(x)}{u_k(x_k)} \cdot \frac{v(x)}{v(x_k)} \cdot \frac{\omega(x)}{\omega(x_k)} f(x_k) +$$

$$+ \sum_{h=1}^{m} \frac{u(x)}{u(t_h)} \cdot \frac{v_h(x)}{v_h(t_h)} \cdot \frac{\omega(x)}{\omega(t_h)} f(t_h) +$$

$$+ \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} \sum_{\nu=0}^{r_i-j-1} \frac{(x - a_i)^j}{j!} \left[\frac{(x - a_i)^\nu}{\nu!} \left(\frac{1}{\omega_i(x)} \right)_{a_i}^{(\nu)} \right] \omega_i(x) f^{(j)}(a_i),$$

where

$$u_k(x) = u(x)/(x - x_k), \quad v_h(x) = v(x)/(x - t_h), \quad \omega_i(x) = \omega(x)/(x - a_i)^{r_i}.$$

3. By integrating the preceding interpolation formula we obtain a quadrature formula of the following form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{h=1}^{m} B_h f(t_h) + \sum_{i=1}^{s} \sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f), \tag{6}$$

where

$$A_k = \int_{-1}^{1} \frac{u_k(x)}{u_k(x_k)} \cdot \frac{v(x)}{v(x_k)} \cdot \frac{\omega(x)}{\omega(x_k)} dx, \tag{7}$$

$$B_h = \int_{-1}^{1} \frac{u(x)}{u(t_h)} \cdot \frac{v_h(x)}{v_h(t_h)} \cdot \frac{\omega(x)}{\omega(t_h)} dx, \tag{8}$$

$$C_{i,j} = \sum_{\nu=0}^{r_i-j-1} \int_{-1}^{1} \frac{(x - a_i)^j}{j!} \left[\frac{(x - a_i)^\nu}{\nu!} \left(\frac{1}{\omega_i(x)} \right)_{a_i}^{(\nu)} \right] \omega_i(x) dx, \tag{9}$$

$$R(f) = \int_{-1}^{1} u(x)v(x)\omega(x) \left[x, \begin{matrix} x_k \\ 1 \end{matrix}, \begin{matrix} t_h \\ 1 \end{matrix}, \begin{matrix} a_i \\ r_i \end{matrix} ; f \right] dx.$$

The brackets used in this remainder represent the symbol for divided differences.

4. Now we want to determine the nodes x_k so that we have $B_h = 0$ ($h = 1, 2, \dots, m$) for any values of the parameters $t_1, t_2, \dots, t_m$. It is easy to see that this is equivalent with the condition that the polynomial $u(x)$ is orthogonal on $(-1, 1)$, with respect to the weight function $\omega(x)$, with any polynomial of degree $m-1$, since $t_1, t_2, \dots, t_m$ are arbitrary numbers.

But it is known [4] that we must have

$$U_m(x) = \begin{vmatrix} L_m(x) & L_{m+1}(x) & \dots & L_{m+p}(x) \\ L_m(a_1) & L_{m+1}(a_1) & \dots & L_{m+p}(a_1) \\ L'_m(a_1) & L'_m(a_1) & \dots & L'_{m+p}(a_1) \\ \dots & \dots & \dots & \dots \\ L_m^{(r_1-1)}(a_1) & L_{m+1}^{(r_1-1)}(a_1) & \dots & L_{m+p}^{(r_1-1)}(a_1) \\ L_m(a_2) & L_{m+1}(a_2) & \dots & L_{m+p}(a_2) \\ \dots & \dots & \dots & \dots \\ L_m(a_s) & L_{m+1}(a_s) & \dots & L_{m+p}(a_s) \\ L'_m(a_s) & L'_{m+1}(a_s) & \dots & L'_{m+p}(a_s) \\ \dots & \dots & \dots & \dots \\ L_m^{(r_s-1)}(a_s) & L_{m+1}^{(r_s-1)}(a_s) & \dots & L_{m+p}^{(r_s-1)}(a_s) \end{vmatrix} : \omega(x), \tag{10}$$

where by L_n we denote the Legendre polynomial of degree n:

$$L_n(x) = (2^n \cdot n!)^{-1}[(x^2-1)^n]^{(n)} \quad \text{and} \quad u(x) = \widetilde{U}_m(x).$$

If we take into consideration the formula (8) for the coefficient B_h, we can see that in order to have $B_1 = \dots = B_m = 0$ it is necessary and sufficient that

$$\int_{-1}^{1} \omega(x)u(x)g(x)dx = 0,$$

where $g(x)$ is any polynomial of degree $m-1$.

But it is known [4] that in this case the node polynomial $u(x)$ can be found by means of the formula (10).

We make the remark that the nodes x_k can be found also by determining the minimum of the following function of m variables

$$F(u_1, \dots, u_m) = \int_{-1}^{1} \omega(x)(x-u_1)^2 \dots (x-u_m)^2 dx. \tag{11}$$

5. Because $t_1, t_2, \dots, t_m$ are arbitrary numbers, we can make $t_k \to x_k$ ($k = 1, 2, \dots, m$).

In this case we arrive at the following quadrature formula of Gauss-Christoffel-Stancu type

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{i=1}^{s}\sum_{j=0}^{r_i-1} C_{i,j} f^{(j)}(a_i) + R(f), \tag{12}$$

where

$$A_k = \int_{-1}^{1} \left(\frac{u_k(x)}{u_k(x_k)}\right)^2 \frac{\omega(x)}{\omega(x_k)} dx$$

and

$$R(f) = \int_{-1}^{1} \omega(x)u^2(x) \left[x, \begin{array}{c} x_k \\ 2 \end{array}, \begin{array}{c} a_i \\ r_i \end{array}; f \right] dx, \quad k = \overline{1,m};\ i = \overline{1,s}.$$

One observes that all the coefficients A_k are positive.

Assuming that $f \in C^{2m+p}(-1,1)$, by using the mean-value theorem of divided differences we can give the following representation of the remainder

$$R(f) = \frac{f^{(2m+p)}(\xi)}{(2m+p)!} \int_{-1}^{1} \omega(x)u^2(x)dx, \quad \xi \in (-1,1). \tag{13}$$

6. Now we make the remark that if the polynomial of the fixed nodes: $\pm a_1, \pm a_2, \dots, \pm a_s$ $(2s = r)$ is even, then we can obtain the following equation for determining the Gaussian nodes x_k:

$$\begin{vmatrix} L_m(x) & L_{m+2}(x) & \dots & L_{m+r}(x) \\ L_m(a_1) & L_{m+2}(a_1) & \dots & L_{m+r}(a_1) \\ L'_m(a_1) & L'_{m+2}(a_1) & \dots & L'_{m+r}(a_1) \\ \dots & \dots & \dots & \dots \\ L_m^{(r_1-1)}(a_1) & L_{m+2}^{(r_1-1)}(a_1) & \dots & L_{m+r}^{(r_1-1)}(a_1) \\ L_m(a_2) & L_{m+2}(a_2) & \dots & L_{m+r}(a_2) \\ \dots & \dots & \dots & \dots \\ L_m(a_s) & L_{m+2}(a_s) & \dots & L_{m+r}(a_s) \\ L'_m(a_s) & L'_{m+2}(a_s) & \dots & L'_{m+r}(a_s) \\ \dots & \dots & \dots & \dots \\ L_m^{(r_s-1)}(a_s) & L_{m+2}^{(r_s-1)}(a_s) & \dots & L_{m+r}^{(r_s-1)}(a_s) \end{vmatrix} = 0, \tag{14}$$

where by $L_n(x)$ we denote again the Legendre orthogonal polynomial of degree n.

7. If we normalize the orthogonal polynomial given at (10), then we obtain

$$\widehat{U}(x) = \frac{1}{\gamma_m} \sqrt{\frac{(-1)^p \beta_m}{\beta_{m+p} G_m G_{m+1}}} \cdot U_m(x),$$

where γ_m is the coefficient of x^m from the Legendre polynomial $L_m(x)$, that is

$$\gamma_m = \int_{-1}^{1} L_m^2(x)dx = \frac{2}{2m+1}$$

and by G_k we denote the following determinant

$$\begin{vmatrix} L_k(a_1) & L_{k+1}(a_1) & \dots & L_{k+p-1}(a_1) \\ L'_k(a_1) & L'_{k+1}(a_1) & \dots & L'_{k+p-1}(a_1) \\ \dots & \dots & \dots & \dots \\ L_k^{(r_1-1)}(a_1) & L_{k+1}^{(r_1-1)}(a_1) & \dots & L_{k+p-1}^{(r_1-1)}(a_1) \\ L_k(a_2) & L_{k+1}(a_2) & \dots & L_{k+p-1}(a_2) \\ L'_k(a_2) & L'_{k+1}(a_2) & \dots & L'_{k+p-1}(a_s) \\ \dots & \dots & \dots & \dots \\ L_k(a_s) & L_{k+1}(a_s) & \dots & L_{k+p-1}(a_s) \\ \dots & \dots & \dots & \dots \\ L_k^{(r_s-1)}(a_s) & L_{k+1}^{(r_s-1)}(a_s) & \dots & L_{k+p-1}^{(r_s-1)}(a_s) \end{vmatrix}.$$

By using the known Christoffel-Darboux formula from the theory of orthogonal polynomials, we can obtain for the coefficients A_k of the quadrature formula (12) the expressions

$$A_k = \int_{-1}^{1} \frac{\widehat{U}_m(t)\omega(t)dt}{(t-x_k)\widehat{U}'_m(x_k)\omega(x_k)} = \frac{1}{\sqrt{\lambda_m}\,\omega(x_k)\widehat{U}'_m(x_k)\widehat{U}_{m-1}(x_k)}.$$

8. In order to present some illustrations we consider that the fixed nodes are: $a_1 = -1$, $a_2 = 1$, having different orders of multiplicities.

If the polynomial of the fixed nodes is $\omega(x) = (1+x)(1-x)^2$, then the Gaussian nodes can be found by solving the equation

$$\begin{vmatrix} L_m(x) & L_{m+1}(x) & L_{m+2}(x) & L_{m+3}(x) \\ L_m(-1) & L_{m+1}(-1) & L_{m+2}(-1) & L_{m+3}(-1) \\ L_m(1) & L_{m+1}(1) & L_{m+2}(1) & L_{m+3}(1) \\ L'_m(1) & L'_{m+1}(1) & L'_{m+2}(1) & L'_{m+3}(1) \end{vmatrix} = 0.$$

It leads to the solution of the equation

$$(2m+5)[L_m(x) - L_{m+2}(x)] - (2m+3)[L_{m+1}(x) - L_{m+3}(x)] = 0,$$

eliminating the roots of the polynomial $\omega(x)$.

If we take $m = 1$ we find the Gaussian node $x_1 = -\frac{1}{5}$ and the quadrature formula of degree of exactness four

$$\int_{-1}^{1} f(x)dx = \frac{1}{108}\left[27f(-1) + 125f\left(-\frac{1}{5}\right) + 64f(1) - 12f'(1)\right] + \frac{2}{1125}f^{(5)}(\xi),$$

given first in [4].

For $m = 2$ we get the Gaussian nodes

$$x_1 = -\frac{2\sqrt{2}+1}{7}, \quad x_2 = \frac{2\sqrt{2}-1}{7}.$$

By using these nodes and the fixed nodes $a_1 = -1$ (simple) and $a_2 = 1$ (double), we can obtain a quadrature formula of degree of exactness six.

If we now consider that $\omega(x) = (1-x^2)^2$ then the Gaussian nodes and the fixed nodes are the roots of the equation

$$(2m+7)L_m(x) + (2m+3)L_{m+1}(x) - 2(2m+5)L_{m+2}(x) = 0.$$

In the case $m = 3$ we obtain a quadrature formula of degree of exactness nine, namely

$$\int_{-1}^{1} f(x)dx = \frac{1}{105}\left[19f(-1) + f'(-1) + 54f\left(-\frac{1}{\sqrt{3}}\right) + 64f(0)+\right.$$
$$\left.+54f\left(\frac{1}{\sqrt{3}}\right) - f'(1) + 19f(1)\right] + \frac{1}{589396500}f^{(10)}(\xi).$$

Considering also the case $\omega(x) = (1-x^2)^3$, formula (10) leads to the solution of the equation

$$(2m+7)(2m+9)(2m+11)L_m(x) - 3(2m+5)(2m+7)(2m+11)L_{m+2}(x)+$$
$$+3(2m+3)(2m+7)(2m+9)L_{m+4}(x) - (2m+3)(2m+5)(2m+7)L_{m+6}(x) = 0.$$

In the case $m = 2$ we obtain the following quadrature formula of degree of exactness nine

$$\int_{-1}^{1} f(x)dx = \frac{1}{3360}\left[1173f(-1) + 156f'(-1) + 8f''(-1) + 2187f\left(-\frac{1}{3}\right) +\right.$$
$$\left.+2187f\left(\frac{1}{3}\right) + 8f''(1) - 156f'(1) + 1173f(1)\right] - \frac{2}{442047375}f^{(10)}(\xi).$$

For $m = 3$ we get the Gaussian nodes

$$x_1 = -\sqrt{\frac{3}{11}}, \quad x_2 = 0, \quad x_3 = \sqrt{\frac{3}{11}}$$

and a Gauss-Christoffel quadrature formula of degree of exactness eleven.

9. Considering that we have an arbitrary real fixed node a, of multiplicity $2s$, we arrive at a quadrature formula of the form

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{m} A_k f(x_k) + \sum_{h=0}^{2s-1} B_h f^{(h)}(a) + R(f),$$

where the remainder has the expression

$$R(f) = \frac{f^{(2m+2s)}(\xi)}{(2m+2s)!}\int_{-1}^{1}(x-a)^{2s}\widehat{U}_m^2(x)dx.$$

The Gaussian nodes can be found by solving the equation

$$\begin{vmatrix} L_m(x) & L_{m+1}(x) & \dots & L_{m+2s}(x) \\ L_m(a) & L_{m+1}(a) & \dots & L_{m+2s}(a) \\ L'_m(a) & L'_{m+1}(a) & \dots & L'_{m+2s}(a) \\ \dots & \dots & \dots & \dots \\ L_m^{(2s-1)}(a) & L_{m+1}^{(2s-1)}(a) & \dots & L_{m+2s}^{(2s-1)}(a) \end{vmatrix} = 0,$$

omitting the root a of multiplicity $2s$.

In the case when $\omega(x) = x^2$ and we take $m = 5$ we find the Gaussian nodes

$$-x_1 = x_4 = \sqrt{\frac{21 + 2\sqrt{14}}{33}}, \quad -x_2 = x_3 = \sqrt{\frac{21 - 2\sqrt{14}}{33}}$$

and we are able to obtain a quadrature formula having the degree of exactness eleven, namely

$$\int_{-1}^{1} f(x)dx = \frac{1}{514500}\{440832f(0) + 8960f''(0)+$$
$$+27(5446 - 537\sqrt{14})[f(x_1) + f(x_4)] + 27(5446 + 537\sqrt{14})[f(x_2) + f(x_3)]\}+$$
$$+\frac{1}{476804928600}f^{(12)}(\xi).$$

Considering also the case $\omega(x) = x^4$ and $m = 3$ we get the Gaussian nodes

$$x_1 = -\frac{\sqrt{7}}{3}, \quad x_2 = 0, \quad x_3 = \frac{\sqrt{7}}{3}$$

and the following quadrature formula

$$\int_{-1}^{1} f(x)dx = \frac{1}{36015}\left\{50160f(0) + 3500f''(0) + 10935\left[f\left(-\frac{\sqrt{7}}{3}\right) + f\left(\frac{\sqrt{7}}{3}\right)\right]\right\}+$$
$$+\frac{1}{404157600}f^{(10)}(\xi).$$

Ending this paper we mention that D. D. Stancu and A. H. Stroud have tabulated the values of the nodes, the coefficients and the remainders, with 20 significant digits, in the paper [6].

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