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THE STUDY OF REMAINDER FOR SOME CUBATURE FORMULAS FOR TRIANGULAR DOMAIN

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The purpose of this paper is to give some practical cubature formulas in approximation of the integral

$$I = \int_D f(x,y)\,dxdy \tag{1}$$

where D is a triangular domain, $D = \{(x,y)/x \geq 0, y \geq 0, x+y \leq h\}$ and $f : D \to R$ is an integrable function on D. We would like to construct some practical cubature formulas of the following form:

$$I = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij} f(x_i, y_j) + R_{mn}(f)$$

where A_{ij} are the coefficients of the formula and $R_{mn}(f)$ the remainder. We will use the quadrature rules given by Bruno Welfer in [2] for triangular domain and we will study the remainder of these rules and some optimal properties. To obtain the error of the approximation formula we will use a generalization of the Peano Theorem, when the function is a member of the so-called Sard space. We will note with B_{pq} the Sard space where $p,q \in N, p+q = m$. Let $\Omega = [0,h] \times [0,h]$ where $h \in R_+$, then the Sard space $B_{p,q}(0,0)$ is the set of all of the functions $f : \Omega \to R$ with the following properties:

1. $f^{(p,q)} \in C(\Omega)$
2. $f^{(m-j,j)}(\cdot, 0) \in C[0,h]$, $j = 0,...,q-1$
3. $f^{(i,m-i)}(0, \cdot) \in C[0,h]$, $i = 0,...,p-1$

Theorem 1. *Let $L : B_{pq}(0,0) \to R$ be a continuous linear functional. If $Ker(L) = P^2_{m-1}$ then*

$$L(f) = \sum_{j<q}\int_0^h K_{m-j,j}(s) f^{(m-j,j)}(s,0)ds + \sum_{i<p} K_{i,m-i}(t) f^{(i,m-i)}(0,t)dt + \\ + \int\int_{\omega} K_{p,q}(s,t) f^{(p,q)}(s,t)dsdt \tag{2}$$

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where

$$K_{m-j,j}(s) = L^{(x,y)}\left[\frac{(x-s)_+^{m-j-1}}{(m-j-1)!}\frac{y^j}{j!}\right], j < q$$

$$K_{i,m-i}(t) = L^{(x,y)}\left[\frac{x^i}{i!}\frac{(y-t)_+^{m-i-1}}{(m-i-1)!}\right], i < p$$

$$K_{p,q}(s,t) = L^{(x,y)}\left[\frac{(x-s)_+^{p-1}}{(p-1)!}\frac{(y-t)_+^{q-1}}{(q-1)!}\right].$$

First of all we consider the cubature formula with a single knots:

$$\int\int_D f(x,y)dxdy = \frac{h^2}{2}f(\frac{h}{3},\frac{h}{3}) + R_1(f). \tag{3}$$

Because the degree of exactness of this formula is equal with 1, we can use the first theorem and follows:

Theorem 2. *If $f^{(2,0)}(\cdot,0) \in C[0,h]$, $f^{(0,2)}(0,\cdot) \in C[0,h]$ and $f^{(1,1)}(x,y) \in C(D)$ then we can give the following delimitation of the error in formula (3):*

$$|R_1(f)| \le \frac{h^4}{72}M_{20}f + \frac{h^4}{72}M_{02}f + \frac{89h^4}{1944}M_{11}f,$$

where

$$M_{20}f = \max_{x\in[0,h]}|f^{(2,0)}(x,0)|, M_{02}f = \max_{y\in[0,h]}|f^{(0,2)}(0,y)|, M_{11}f = \max_D|f^{(1,1)}(x,y)|.$$

Proof. Theorem 1 implies the following error representation:

$$R_1(f) = \int_0^h K_{20}(s)f^{(2,0)}(s,0)ds + \int_0^h K_{02}(s)f^{(0,2)}(0,t)dt$$
$$+ \int\int_{T_h} K_{11}(s,t)f^{(1,1)}(s,t)dsdt \tag{4}$$

and

$$K_{20}(s) = R^{xy}\left[(x-s)_+\right]$$
$$K_{02}(t) = R^{xy}\left[(y-t)_+\right]$$
$$K_{11}(s,t) = R^{xy}\left[(x-s)_+^0(y-t)_+^0\right].$$

Therefore the so-called Peano-kernels has the representation

$$K_{20}(s) = \begin{cases} \frac{(h-s)^3}{6} - \frac{h^2}{2}(\frac{h}{3}-s), & s \le \frac{h}{3} \\ \frac{(h-s)^3}{6}, & s \ge \frac{h}{3} \end{cases}$$

$$K_{02}(t) = \begin{cases} \frac{(h-t)^3}{6} - \frac{h^2}{2}(\frac{h}{3}-t), & t \le \frac{h}{3} \\ \frac{(h-t)^3}{6}, & t \ge \frac{h}{3} \end{cases}$$

and

$$K_{11}(s,t) = \begin{cases} \frac{(h-t-s)^3}{6} - \frac{h^2}{2}, & 0 \le s,t \le \frac{h}{3} \\ \frac{(h-t-s)^3}{6}, & 0 \le s \le \frac{h}{3}, \frac{h}{3} \le t \le h \ \ or \ \ \frac{h}{3} \le s \le h, 0 \le t \le \frac{h}{3} \end{cases}$$

If we study the sign of these functions, we conclude that K_{20} and K_{02} are positive functions at the interval $[0,h]$ and $K_{11}(s,t) \le 0, 0 \le s,t \le \frac{h}{3}$ and $K_{11}(s,t) \ge 0 \ \ if \ \ 0 \le s \le \frac{h}{3}, \frac{h}{3} \le t \le h \ \ or \ \ \frac{h}{3} \le s \le h, 0 \le t \le \frac{h}{3}$.

Since

$$\int_0^h K_{20}(s)ds = \frac{h^4}{72},$$

$$\int_0^h K_{02}(t)dt = \frac{h^4}{72},$$

and

$$\int\int_D |K_{11}(s,t)|\, dsdt = \frac{89h^4}{1944},$$

from (4) finally yields the theorem.

Let now consider the following formula:

$$\int_D\int f(x,y)dxdy = \frac{h^2}{6}\left[f(0,\frac{h}{2}) + f(\frac{h}{2},0) + f(\frac{h}{2},\frac{h}{2})\right] + R_2(f). \tag{5}$$

The degree of exactness of this formula is 2, therefore we can use the theorem 1 for the representation of the error, an we can give the following delimitation of the approximation error:

Theorem 3. *If* $f^{(3,0)}(\cdot,0) \in C[0,h], f^{(2,1)}(\cdot,0) \in C[0,h], f^{(0,3)}(0,\cdot) \in C[0,h]$ *and* $f^{(1,2)}(s,t) \in C(D)$ *than we have*

$$|R_2(f)| \le M_{30}f\frac{h^5}{720} + M_{21}f\frac{h^5}{364} + M_{03}f\frac{h^5}{720} + M_{12}f\frac{h^5}{24} \tag{6}$$

where

$$M_{30}f = \max_{s\in[0,h]}\left|f^{(3,0)}(s,0)\right|, M_{21}f = \max_{s\in[0,h]}\left|f^{(2,1)}(s,0)\right|,$$

$$M_{03}f = \max_{t\in[0,h]}\left|f^{(0,3)}(0,t)\right| \ \ and \ M_{12}f = \max_D\left|f^{(1,2)}(s,t)\right|.$$

Proof. We will use the same method like in the previous theorem, than the error of the formula (5) is

$$R_2(f) = \int_0^h K_{30}(s)f^{(3,0)}(s)ds + \int_0^h K_{21}(s)f^{(2,1)}(s,0)ds + \int_0^h K_{03}(t)f^{(0,3)}(0,t)dt+$$
$$+\int_D\int K_{12}(s,t)f^{(1,2)}(s,t)dsdt$$

where

$$K_{30}(s) = \begin{cases} \frac{(h-s)^4}{24} - \frac{h^3}{6}(\frac{h}{2} - s)^2, & s < h/2 \\ \frac{(h-s)^4}{24}, & s \ge h/2 \end{cases}$$

$$K_{21}(s) = \begin{cases} \frac{(h-s)^4}{24} - \frac{h^3}{12}(\frac{h}{2} - s), & s < h/2 \\ \frac{(h-s)^4}{24}, & s \geq h/2 \end{cases}$$

$$K_{12}(s,t) = \begin{cases} \frac{(h-s-t)^3}{6} - \frac{h^2}{6}(\frac{h}{2} - t), & 0 < s,t < h/2 \\ \frac{(h-s-t)^3}{6}, & otherwise \end{cases}$$

and

$$K_{03}(t) = \begin{cases} \frac{(h-t)^4}{24} - \frac{h^2}{6}(\frac{h}{2} - t)^2, & t < h/2 \\ \frac{(h-t)^4}{24}, & t \geq h/2 \end{cases}$$

The kernel functions K_{30} and K_{03} are positive on the interval $[0,h]$ and their integral on the same interval is equal with $\frac{h^5}{720}$. Also we have $\max\limits_{s\in[0,h]} |K_{21}(x,y,s)| = \frac{h^4}{384}$ and $\max\limits_{(s,t)\in D} |K_{12}(x,y,s,t)| = \frac{h^3}{12}$, therefore we can give the following delimitation of the absolute error:

$$\begin{aligned} |R_2(f)| &\leq \left| \int_0^h K_{30}(s) f^{(3,0)}(s,0)ds \right| + \left| \int_0^h K_{21}(s) f^{(2,1)}(s,0)ds \right| + \\ &+ \left| \int_0^h K_{03}(t) f^{(0,3)}(0,t)dt \right| + \left| \int_D\int K_{12}(s,t) f^{(1,2)}(s,t)dsdt \right| \\ &\leq M_{30}f \int_0^h |K_{30}(s)|\, ds + \frac{h^4}{364} \int_0^h \left| f^{(2,1)}(s,0) \right| ds + \\ &+ M_{03}f \int_0^h |K_{03}(t)|\, dt + \frac{h^3}{12} \int_D\int \left| f^{(1,2)}(s,t) \right| dsdt \\ &\leq M_{30}f \frac{h^5}{720} + M_{21}f \frac{h^5}{364} + M_{03}f \frac{h^5}{720} + M_{12}f \frac{h^5}{24}. \end{aligned}$$

Finally we consider the following cubature formula:

$$\begin{aligned} \int_D\int f(x,y)dxdy &= \frac{h^2}{120} \Big[3f(0,0) + 3f(h,0) + 3f(0,h) + 8f(\frac{h}{2},0) + \\ &+ 8f(\frac{h}{2},\frac{h}{2}) + 8f(0,\frac{h}{2}) + 27f(\frac{h}{3},\frac{h}{3}) \Big] + R_3(f). \end{aligned}$$

Because the degree of exactness of this formula is equal with 3, we can give the following theorem for the delimitation of the absolute error:

Theorem 4. *If* $f^{(4,0)}(s,0) \in C[0,h]$, $f^{(3,1)}(s,0) \in C[0,h]$, $f^{(0,4)}(0,t) \in C[0,h]$, $f^{(1,3)}(0,t) \in C[0,h]$ *and* $f^{(2,2)}(s,t) \in C(D)$ *then*

$$|R_3(f)| \leq M_{40}f \frac{h^6}{8640} + M_{31}f \frac{7h^6}{1440} + M_{13}f \frac{7h^6}{1440} + M_{04}f \frac{h^6}{8640} + M_{22}f \frac{h^6}{768},$$

where

$$M_{40}f = \max_{s\in[0,h]} \left| f^{(4,0)}(s,0) \right|, M_{31}f = \max_{s\in[0,h]} \left| f^{(3,1)}(s,0) \right|, M_{13}f = \max_{t\in[0,h]} \left| f^{(1,3)}(0,t) \right|,$$

$$M_{04}f=\max_{t\in[0,h]}\left|f^{(0,4)}(0,t)\right|, M_{22}f=\max_{(s,t)\in D}\left|f^{(2,2)}(s,t)\right|.$$

Proof. We will use again the generalization of the Peano theorem in bidimensional case and we have

$$R_3(f)=\int_0^h K_{40}(s)f^{(4,0)}(s,0)ds+\int_0^h K_{31}(s)f^{(3,1)}(s,0)ds+\int_0^h K_{04}(t)f^{(0,4)}(0,t)dt+$$
$$+\int_0^h K_{13}(t)f^{(1,3)}(0,t)dt+\int_D\int K_{22}(s,t)f^{(2,2)}(s,t)dsdt$$

where the K_{40}, K_{31}, K_{13} and K_{22} are the so-called kernel functions, and they have the following representation:

$$K_{40}(s)=\begin{cases}\frac{(h-s)^5}{5!}-\frac{h^2}{120}\left[\frac{(h-s)^3}{2}+8\frac{(\frac{h}{2}-s)^3}{3}+9\frac{(\frac{h}{3}-s)^3}{2}\right], & s\in\left[0,\frac{h}{3}\right]\\ \frac{(h-s)^5}{5!}-\frac{h^2}{120}\left[\frac{(h-s)^3}{2}+8\frac{(\frac{h}{2}-s)^3}{3}\right], & s\in(\frac{h}{3},\frac{h}{2})\\ \frac{(h-s)^5}{5!}-\frac{h^2}{120}\frac{(h-s)^3}{2}, & s\in\left[\frac{h}{2},h\right]\end{cases}$$

$$K_{31}(s)=\begin{cases}-\frac{(h-s)^5}{5!}-\frac{h^2}{120}\left[h\frac{(h-2s)^2}{2}+h\frac{(h-3s)^2}{2}\right], & s\in\left[0,\frac{h}{3}\right]\\ -\frac{(h-s)^5}{5!}-\frac{h^3}{240}(h-2s)^2, & s\in(\frac{h}{3},\frac{h}{2})\\ -\frac{(h-s)^5}{5!} & s\in\left[\frac{h}{2},h\right]\end{cases}$$

$$K_{13}(t)=\begin{cases}-\frac{(h-t)^5}{5!}-\frac{h^2}{120}\left[h\frac{(h-2t)^2}{2}+h\frac{(h-3t)^2}{2}\right], & t\in\left[0,\frac{h}{3}\right]\\ -\frac{(h-t)^5}{5!}-\frac{h^3}{240}(h-2t)^2, & t\in(\frac{h}{3},\frac{h}{2})\\ -\frac{(h-t)^5}{5!} & t\in\left[\frac{h}{2},h\right]\end{cases}$$

and

$$K_{22}(s,t)=$$

$$=\begin{cases}\frac{(h-s-t)^4}{4!}-\frac{h^2}{120}\left[8(\frac{h}{2}-s)(\frac{h}{2}-t)+27(\frac{h}{3}-s)(\frac{h}{3}-t)\right], & 0\le s,t\le\frac{h}{3}\\ \frac{(h-s-t)^4}{4!}-\frac{h^2}{120}8(\frac{h}{2}-s)(\frac{h}{2}-t) & \frac{h}{3}\le s\le\frac{h}{2}, 0\le t\le\frac{h}{2},\\ & 0\le s\le\frac{h}{3}, \frac{h}{3}\le t\le\frac{h}{2}\\ \frac{(h-s-t)^4}{4!}, & 0\le s\le\frac{h}{2}, \frac{h}{2}\le t\le h,\\ & \frac{h}{2}\le s\le h, 0\le t\le\frac{h}{2}\end{cases}$$

The K_{40}, K_{31} and the K_{13} functions do not change their sign at the interval $[0,h]$ and if we calculate the maximum of the function K_{22} we obtain $\max\limits_D|K_{22}(s,t)|=$

$\frac{h^4}{384}$, therefore for the absolute value of the error we have the following delimitation:

$$\begin{aligned}|R_3(f)| \leq &\left|\int_0^h K_{40}(s)f^{(4,0)}(s,0)ds\right| + \left|\int_0^h K_{31}(s)f^{(3,1)}(s,0)ds\right| + \\ &+\left|\int_0^h K_{04}(t)f^{(0,4)}(0,t)dt\right| + \left|\int_0^h K_{13}(t)f^{(1,3)}(0,t)dt\right| + \\ &+\left|\int_D\int K_{22}(s,t)f^{(2,2)}(s,t)dsdt\right| \\ &\leq M_{40}f\int_0^h |K_{40}(s)|\, ds + M_{31}f\int_0^h |K_{31}(s)|\, ds + M_{04}f\int_0^h |K_{04}(t)|\, dt + \\ &+ M_{13}f\int_0^h |K_{13}(t)|\, dt + M_{22}f\frac{h^6}{768} \\ &= M_{40}f\frac{h^6}{8640} + M_{31}f\frac{7h^6}{1440} + M_{04}f\frac{7h^6}{1440} + M_{04}f\frac{h^6}{1440} + M_{22}f\frac{h^6}{768}.\end{aligned}$$

Remark 1. *The cubature formula (5) has an optimal character, because it satisfy the conditions established by Stroud in* [5] *regarding the minimal number of knots for a cubature formula. If the degree of exactness of a cubature formula is equal with* 2 *then the minimal number of the knots is* $N = n+1$ *where* n *is the dimension number. The cubature formula (5) with the degree of exactness* 2 *and three knots, has a minimal number of knots.*

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