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CORRIGENDUM: ON THE IRRATIONALITY OF SOME ALTERNATING SERIES

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The aim of this note is to point out that Theorem 1 of the first author's paper [1] is incorrect, and to replace it with Theorem A below and give an application.

Theorem 1. *Let* (a_n) *be a sequence of positive integers such that* $a_n(a_1a_2\dots a_{n-1})^2 \to \infty$ *as* $n \to \infty$. *Then the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{a_n(a_1\dots a_{n-1})^2}$ *is irrational.*

The constant sequence $(a_n) = 2, 2, \dots$ is a counterexample. The mistake in the proof lies in assuming that, with $u_i = (a_1 \dots a_{i-1})^{-2}$ and $v_i = a_i$, the sum $\sum_{i=1}^{n}(-1)^{i-1}u_i/v_i$ is a rational number with denominator $v_1 \dots v_n$. In fact, the denominator is $v_n(v_1 \dots v_{n-1})^2$.

The following result is a generalization of Lemma 1 in [1].

Theorem A. *Let* $(r_n) = (h_n/k_n)$ *be a sequence of rational numbers, with* $k_n > 0$, *satisfying*

(i) $r_2 < r_4 < r_6 < \dots < r_5 < r_3 < r_1$ *and*

(ii) $\liminf_{n\to\infty} k_n|r_{n+1} - r_n| = 0$. *Then the alternating series*

$$r_1 - (r_1 - r_2) + (r_3 - r_2) - (r_3 - r_4) + \dots$$

converges and its sum is irrational.

Proof. It follows from (i) and (ii) that the conditions of Leibniz's alternating series test are satisfied. Thus the series converges and its sum, θ, lies between the partial sums r_n and r_{n+1}, for $n = 1, 2, \dots$ Suppose now that $\theta = a/b$ is rational, $b > 0$. Then (ii) and the inequalities $0 < |\theta - r_n| < |r_{n+1} - r_n|$ imply that $0 < |ak_n - bh_n| < bk_n|r_{n+1} - r_n| < 1$, for some $n \geq 1$. This contradicts the fact that $ak_n - bh_n$ is an integer, completing the proof. □

As an application of Theorem A (or of Lemma 1), we obtain a new proof that *if* p_n/q_n *is the* n*-th convergent of an infinite simple continued fraction,* $n = 0, 1, 2, \dots$, *then the sum of the series* $p_0/q_0 + \sum_{n=0}^{\infty}(-1)^n/(q_nq_{n+1})$ *is an irrational number,* namely, the value of the continued fraction.

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References

[1] J. Sándor, *On the irrationality of some alternating series*, Studia Univ. Babeş-Bolyai, **33**(1988), 8-12.

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