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# ON THE DIRECT LIMIT OF A DIRECT SYSTEM OF COMPLETE MULTIALGEBRAS

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**Abstract**. In this paper we will prove that the direct limit of a direct system of complete multialgebras is a complete algebra.

## 1. Introduction

This paper deals with multialgebras. An important instrument in the study of the multialgebras is fundamental relation of a multialgebra, which can bring us into the class of the universal algebras. In [9] we proved that the fundamental algebra of a multialgebra verifies the identities of the given multialgebra. When trying to obtain multialgebras that verify (even in a weak manner) the identities of their fundamental algebra we obtained a new class of multialgebras. In the particular case of the semihypergroups these multialgebras are the complete semihypergroups that is why we called this multialgebras complete. We will prove that the complete multialgebras form a class of multialgebras closed under the formation of the direct limits of direct systems.

## 2. Preliminaries

Let $\tau = (n_\gamma)_{\gamma<o(\tau)}$ be a sequence with $n_\gamma \in \mathbb{N} = \{0, 1, \ldots\}$, where $o(\tau)$ is an ordinal and for any $\gamma < o(\tau)$, let $\mathbf{f}_\gamma$ be a symbol of an $n_\gamma$-ary (multi)operation and let us consider the algebra of the $n$-ary terms (of type $\tau$) $\mathfrak{P}^{(n)}(\tau) = (\mathbf{P}^{(n)}(\tau), (f_\gamma)_{\gamma<o(\tau)})$.

Let $A$ be a set and $P^*(A)$ the set of the nonempty subsets of $A$. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<o(\tau)})$ be a multialgebra, where, for any $\gamma < o(\tau)$, $f_\gamma : A^{n_\gamma} \to P^*(A)$ is the multioperation of arity $n_\gamma$ that corresponds to the symbol $\mathbf{f}_\gamma$. One can admit that the support set $A$ of the multialgebra $\mathfrak{A}$ is empty if there are no nullary multioperations among the multioperations $f_\gamma,\ \gamma < o(\tau)$. Of course, any universal algebra is a multialgebra (we can identify an one element set with its element).

Defining for any $\gamma < o(\tau)$ and for any $A_0, \ldots, A_{n_\gamma-1} \in P^*(A)$

$$f_\gamma(A_0, \ldots, A_{n_\gamma-1}) = \bigcup\{f_\gamma(a_0, \ldots, a_{n_\gamma-1}) \mid\ a_i \in A_i,\ i \in \{0, \ldots, n_\gamma - 1\}\},$$

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we obtain a universal algebra on $P^*(A)$ (see [11]). We denote this algebra by $\mathfrak{P}^*(\mathfrak{A})$. As in [6], we can construct, for any $n \in \mathbb{N}$, the algebra $\mathfrak{P}^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ of the $n$-ary term functions on $\mathfrak{P}^*(\mathfrak{A})$.

A mapping $h : A \to B$ between the multialgebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type $\tau$ is called homomorphism if for any $\gamma < o(\tau)$ and for all $a_0, \dots, a_{n_\gamma - 1} \in A$ we have

$$h(f_\gamma(a_0, \dots, a_{n_\gamma-1})) \subseteq f_\gamma(h(a_0), \dots, h(a_{n_\gamma-1})). \tag{1}$$

A bijective mapping $h$ is a multialgebra isomorphism if both $h$ and $h^{-1}$ are multialgebra homomorphisms. The multialgebra isomorphisms can also be characterized as being those bijective homomorphisms for which (1) holds with equality.

**Proposition 1.** [8, Proposition 1] *For a homomorphism $h : A \to B$, if $n \in \mathbb{N}$, $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ and $a_0, \dots, a_{n-1} \in A$ then $h(p(a_0, \dots, a_{n-1})) \subseteq p(h(a_0), \dots, h(a_{n-1}))$.*

The *fundamental relation* of a multialgebra $\mathfrak{A}$ as the transitive closure $\alpha^*$ of the relation $\alpha$ given on $A$ as follows: for $x, y \in A$, $x\alpha y$ if and only if $x, y \in p(a_0, \dots, a_{n-1})$ for some $n \in \mathbb{N}$, $p \in P^{(n)}(\mathfrak{P}^*(\mathfrak{A}))$ and $a_0, \dots, a_{n-1} \in A$ (see [7] and [9]). The relation $\alpha^*$ is the smallest equivalence relation on $A$ such that the factor multialgebra $\mathfrak{A}/\alpha^*$ is a universal algebra. We denoted the class of $a \in A$ modulo $\alpha^*$ by $\overline{a}$ and $A/\alpha^*$ by $\overline{A}$. We also denoted the algebra $\mathfrak{A}/\alpha^*$ by $\overline{\mathfrak{A}}$ and we called it the *fundamental algebra* of the multialgebra $\mathfrak{A}$.

**Proposition 2.** [9, Proposition 3] *The following conditions are equivalent for a multialgebra $\mathfrak{A} = (A, (f_\gamma)_{\gamma < o(\tau)})$ of type $\tau$ :*

(i) *for all $\gamma < o(\tau)$, for all $a_0, \dots, a_{n_\gamma-1} \in A$,*

$$a \in f_\gamma(a_0, \dots, a_{n_\gamma-1}) \;\Rightarrow\; \overline{a} = f_\gamma(a_0, \dots, a_{n_\gamma-1}).$$

(ii) *for all $m \in \mathbb{N}$, for all $\mathbf{q}, \mathbf{r} \in P^{(m)}(\tau) \setminus \{\mathbf{x}_i \mid i \in \{0, \dots, m-1\}\}$, for all $a_0, \dots, a_{m-1}, b_0, \dots, b_{m-1} \in A$,*

$$q(a_0, \dots, a_{m-1}) \cap r(b_0, \dots, b_{m-1}) \neq \emptyset \;\Rightarrow\; q(a_0, \dots, a_{m-1}) = r(b_0, \dots, b_{m-1}).$$

The multialgebras which verify one of the equivalent conditions (i) and (ii) from the previous proposition are generalizations for the complete semihypergroups (see [3, Definition 137]). This fact suggests the following:

**Definition 1.** A multialgebra which satisfies one of the equivalent conditions from the previous proposition will be called *complete multialgebra*.

*Remark* 1. As we have seen in [9], a hypergroup $(H, \circ)$ can be identified with a multialgebra $(H, \circ, /, \backslash)$ with three binary multioperations, with $H \neq \emptyset$, $\circ$ associative (i.e. $(a \circ b) \circ c = a \circ (b \circ c)$, for all $a, b, c \in H$) and

$$a/b = \{x \in H \mid a \in x \circ b\},\; b\backslash a = \{x \in H \mid a \in b \circ x\}. \tag{2}$$

*Remark* 2. In [10] the complete hypergroups are defined as the complete semihypergroups which are hypergroups. For any elements $a$ and $b$ from a complete hypergroup $(H, \circ)$ there exists $b' \in H$ such that $a/b = a \circ b'$ and $b\backslash a = b' \circ a$ (see [10, Theorem 146]). Thus, a hypergroup $(H, \circ)$ is complete if and only if the multialgebra $(H, \circ, /, \backslash)$ from the previous remark is a complete multialgebra.

One can construct the category of the multialgebras of the same type $\tau$ where the morphisms are the multialgebra homomorphisms and the product is the usual mapping composition. We will denote this category by $\mathbf{Malg}(\tau)$. The complete multialgebras form a full subcategory $\mathbf{CMalg}(\tau)$ of the category $\mathbf{Malg}(\tau)$.

## 3. **The direct limit of a direct system of complete multialgebras**

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of complete multialgebras and let multialgebra $\mathfrak{A}_\infty = (A_\infty, (f_\gamma)_{\gamma < o(\tau)})$ be the direct limit of the direct system $\mathcal{A}$.

Remind that $(I, \leq)$ is a directed preordered set and the mappings $\varphi_{ij}$ $(i, j \in I,\ i \leq j)$ are such that for any $i, j, k \in I$, with $i \leq j \leq k$, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ and $\varphi_{ii} = 1_{A_i}$, for all $i \in I$. Also remind that the set $A_\infty$ is the direct limit of the direct system of sets $((A_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ and it is obtained as follows: on the disjoint union $A$ of the sets $A_i$ one defines the relation $\equiv$ as follows: for any $x, y \in A$ there exist $i, j \in I$, such that $x \in A_i,\ y \in A_j$, and $x \equiv y$ if and only if $\varphi_{ik}(x) = \varphi_{jk}(y)$ for some $k \in I$ with $i \leq k,\ j \leq k$. This relation on $A$ is an equivalence and $A_\infty$ is the quotient set $A/_\equiv = \{\widehat{x} \mid x \in A\}$ (see [6]).

The multioperations of the direct limit multialgebra are defined as follows: let $\gamma < o(\tau)$ and $\widehat{x_0}, \ldots, \widehat{x_{n_\gamma - 1}} \in A_\infty$ and for any $j \in \{0, \ldots, n_\gamma - 1\}$ let us consider that $x_j \in A_{i_j}$ $(i_j \in I)$. Then

$$f_\gamma(\widehat{a_0}, \ldots, \widehat{a_{n_\gamma - 1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I,\ i_0 \leq m, \ldots,\ i_{n_\gamma - 1} \leq m,\ a \in f_\gamma(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n_\gamma - 1} m}(a_{n_\gamma - 1}))\}.$$

**Lemma 1.** [10, Lemma 2] *Let* $\mathbf{p} \in \mathbf{P}^{(n)}(\tau)$ *and* $a_0, \ldots, a_{n-1} \in A$. *If* $i_0, \ldots, i_{n-1} \in I$ *are such that* $a_j \in A_{i_j}$ *for all* $j \in \{0, \ldots, n-1\}$ *then*

$$p(\widehat{a_0}, \ldots, \widehat{a_{n-1}}) = \{\widehat{a} \in A_\infty \mid \exists m \in I,\ i_0 \leq m, \ldots,\ i_{n-1} \leq m,\ a \in p(\varphi_{i_0 m}(a_0), \ldots, \varphi_{i_{n-1} m}(a_{n-1}))\}.$$

Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ be a direct system of multialgebras and let us consider $J \subseteq I$ such that $(J, \leq)$ is also directed. We will denote by $\mathcal{A}_J$ the direct system whose carrier is $(J, \leq)$, consisting of the multialgebras $(\mathfrak{A}_i \mid i \in J)$ and the homomorphisms $(\varphi_{ij} \mid i, j \in J,\ i \leq j)$.

**Proposition 3.** [10, Proposition 1] *Let* $\mathcal{A}$ *be a direct system of multialgebras with the carrier* $(I, \leq)$ *and let us consider* $J \subseteq I$ *such that* $(J, \leq)$ *is a directed preordered set cofinal with* $(I, \leq)$. *Then the multialgebras* $\varinjlim \mathcal{A}$ *and* $\varinjlim \mathcal{A}_J$ *are isomorphic.*

*Remark* 3. This proposition was proved for the case when $(I, \leq)$ is an ordered set. Yet, the antisymmetry of the relation $\leq$ is not involved in the proof.

The main result of this paper is the following:

**Theorem 1.** *The category* $\mathbf{CMalg}(\tau)$ *is a subcategory of the category* $\mathbf{Malg}(\tau)$ *which closed under direct limits of direct systems.*

*Proof.* Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i, j \in I,\ i \leq j))$ a direct system of complete multialgebras, let $n \in \mathbb{N}$, $\mathbf{q}, \mathbf{r} \in \mathbf{P}^{(n)}(\tau) \setminus \{\mathbf{x}_j \mid j \in \{0, \ldots, n-1\}\}$ and $\widehat{a_0}, \ldots, \widehat{a_{n-1}}$,

$\widehat{b_0},\dots,\widehat{b_{n-1}}\in A_\infty$. We can consider that the representatives $a_0,\dots, a_{n-1}$, $b_0,\dots, b_{n-1}$ of the given classes are from the set $A_k$ $(k\in I)$. If

$$q(\widehat{a_0},\dots,\widehat{a_{n-1}})\cap r(\widehat{b_0},\dots,\widehat{b_{n-1}})\neq\emptyset$$

then there exists $a\in\bigcup_{i\in I}A_i$ such that

$$\widehat{a}\in q(\widehat{a_0},\dots,\widehat{a_{n-1}})\cap r(\widehat{b_0},\dots,\widehat{b_{n-1}}).$$

From $\widehat{a}\in q(\widehat{a_0},\dots,\widehat{a_{n-1}})$ it results that there exist $m'\in I$, $m'\geq k$, and $a'\equiv a$ such that

$$a'\in q(\varphi_{km'}(a_0),\dots,\varphi_{km'}(a_{n-1}))\subseteq A_{m'}.$$

Analogously, from $\widehat{a}\in r(\widehat{b_0},\dots,\widehat{b_{n-1}})$ it follows that there exist $m''\in I$, $m''\geq k$, and $a''\equiv a$ such that

$$a''\in r(\varphi_{km''}(b_0),\dots,\varphi_{km''}(b_{n-1}))\subseteq A_{m''}.$$

Let $\widehat{x}$ be an arbitrary element from $q(\widehat{a_0},\dots,\widehat{a_{n-1}})$. Then there exists $l\in I$ with $k\leq l$ such that

$$x\in q(\varphi_{kl}(a_0),\dots,\varphi_{kl}(a_{n-1}))\subseteq A_l.$$

From $a'\equiv a\equiv a''$ we deduce the existence of an element $m'''\in I$ with $m'\leq m'''$, $m''\leq m'''$, such that $\varphi_{m'm'''}(a')=\varphi_{m''m'''}(a'')$. Since $(I,\leq)$ is directed, there exists $m\in I$ with $m'''\leq m$ and $l\leq m$. According to Proposition 1 we have

$$\begin{aligned}\varphi_{lm}(x)&\in\varphi_{lm}(q(\varphi_{kl}(a_0),\dots,\varphi_{kl}(a_{n-1})))\\&\subseteq q(\varphi_{lm}(\varphi_{kl}(a_0)),\dots,\varphi_{lm}(\varphi_{kl}(a_{n-1})))\\&=q(\varphi_{km}(a_0),\dots,\varphi_{km}(a_{n-1}))\subseteq A_m.\end{aligned}$$

Also,

$$\begin{aligned}\varphi_{m'm}(a')&\in\varphi_{m'm}(q(\varphi_{km'}(a_0),\dots,\varphi_{km'}(a_{n-1})))\\&\subseteq q(\varphi_{m'm}(\varphi_{km'}(a_0)),\dots,\varphi_{m'm}(\varphi_{km'}(a_{n-1})))\\&=q(\varphi_{km}(a_0),\dots,\varphi_{km}(a_{n-1}))\subseteq A_m\end{aligned}$$

and, analogously,

$$\varphi_{m''m}(a'')\in r(\varphi_{km}(b_0),\dots,\varphi_{km}(b_{n-1}))\subseteq A_m.$$

But

$$\varphi_{m'm}(a')=\varphi_{m'''m}(\varphi_{m'm'''}(a'))=\varphi_{m'''m}(\varphi_{m''m'''}(a''))=\varphi_{m''m}(a''),$$

and, since the multialgebra $\mathfrak{A}_m$ is complete it follows that

$$\varphi_{lm}(x)\in q(\varphi_{km}(a_0),\dots,\varphi_{km}(a_{n-1}))=r(\varphi_{km}(b_0),\dots,\varphi_{km}(b_{n-1})).$$

Consequently, $\widehat{x}\in r(\widehat{b_0},\dots,\widehat{b_{n-1}})$. Thus we have proved that

$$q(\widehat{a_0},\dots,\widehat{a_{n-1}})\subseteq r(\widehat{b_0},\dots,\widehat{b_{n-1}}).$$

Similarly, one can show that $q(\widehat{a_0},\dots,\widehat{a_{n-1}})\supseteq r(\widehat{b_0},\dots,\widehat{b_{n-1}})$, so, we have

$$q(\widehat{a_0},\dots,\widehat{a_{n-1}})=r(\widehat{b_0},\dots,\widehat{b_{n-1}}).$$

Thus the multialgebra $\mathfrak{A}_\infty$ is complete. □

**Corollary 1.** *Let $\mathcal{A} = ((\mathfrak{A}_i \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \le j))$ be a direct system of multialgebras. If any $i,j \in I$ have an upper bound $k \in I$ such that $\mathfrak{A}_k$ is a complete multialgebra, then $\mathfrak{A}_\infty$ is a complete multialgebra.*

This follows from the previous theorem and Proposition 3 since the set

$$J = \{k \in I \mid \mathfrak{A}_k \text{ is a complete multialgebra}\}$$

(with the restriction of $\le$ from $I$) is a directed preordered set cofinal with $(I, \le)$.

In [12] are proved the following theorems:

**Theorem 2.** [12, Theorem 3] *Let $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \le j))$ be a direct system of semihypergroups. The direct limit $(H', \circ)$ of this system is a semihypergroup.*

**Theorem 3.** [12, Theorem 4] *Let $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \le j))$ be a direct system of semihypergroups having the property that for any $i,j \in I$ there exists $k \in I$, $i \le k$, $j \le k$ such that $H_k$ is a hypergroup. The direct limit $(H', \circ)$ of this system is a hypergroup.*

*Remark* 4. In [12] is considered that $(I, \le)$ is partially ordered, but the property holds even if $(I, \le)$ is only preordered.

*Remark* 5. If we see each hypergroup $(H_i, \circ_i)$ as a multialgebra $(H_i, \circ_i, /_i, \backslash_i)$ as in Remark 1 we obtain a direct system of multialgebras of type $\tau$. If we consider for this system the direct limit multialgebra $(H_\infty, \circ_\infty, /, \backslash)$ then $H' = H_\infty$, $\circ = \circ_\infty$ and the multioperations $/, \backslash$ are obtained from $\circ$ using (2).

From Remark 2, Theorem 3 and Theorem 1 we have:

**Corollary 2.** *The direct limit of a direct system of complete (semi)hypergroups is a complete (semi)hypergroup.*

Using, in addition, Corollary 1 we also have:

**Corollary 3.** *Let $(((H_i, \circ_i) \mid i \in I), (\varphi_{ij} \mid i,j \in I,\ i \le j))$ be a direct system of semihypergroups having the property that for any $i,j \in I$ there exists $k \in I$, $i \le k$, $j \le k$ such that $H_k$ is a complete (semi)hypergroup. The direct limit $(H', \circ)$ of this system is a complete (semi)hypergroup.*

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