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UNIVALENCE CRITERIA CONNECTED WITH ARITHMETIC AND GEOMETRIC MEANS

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Abstract. In this paper we obtain some univalence criteria connected with arithmetic and geometric means of the expressions f/g and f'/g', where f and g are analytic functions in the unit disk.

1. Introduction

We let $U_r = \{ z \in C : |z| < r \}$ denote the disk of z-plane, where $r \in (0,1]$, $U_1 = U$ and $I = [0,\infty)$. Let A be the class of functions f analytic in U such that $f(0) = 0,\ f'(0) = 1$. Our consideration apply the theory of Löwner chains; we first recall here the basic result of this theory, from Pommerenke.

Theorem 1.1. ([4]). *Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots,\ a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies the condition $Rep(z,t) > 0$ for all $z \in U,\ t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

2. Univalence criteria and arithmetic mean

In this section we derive several interesting criteria of univalence related to arithmetic mean. The method of prove is based on Theorem 1.1 and on construction of a suitable Löwner chain.

Theorem 2.1. *Let $\alpha,\ \beta,\ \gamma$ be complex numbers such that $\alpha \neq -1$,*

$$|\alpha - \beta| \leq |\beta + 1|, \quad |\gamma - 1| < 1, \quad |\gamma(\alpha + 1) - (\beta + 1)| \leq |\beta + 1|,$$

and let $f \in A$. If there exists a function $g \in A$ such that the inequalities

$$\left| \gamma(\alpha+1)g'(z) - 1 - \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)} \right| < \left| 1 + \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)} \right| \tag{1}$$

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and

$$\left| \left[\gamma(\alpha+1)g'(z) - 1 - \beta\frac{f(z)g'(z)}{f'(z)g(z)}\right]|z|^2 + (1-|z|^2)\left[(\gamma-1)\left(1+\beta\frac{f(z)g'(z)}{f'(z)g(z)}\right)\right.\right.$$

$$\left.\left. + \frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} + \beta\frac{zg'(z)}{g(z)}\left(1-\frac{f(z)g'(z)}{f'(z)g(z)}\right)\right]\right| \le \left|1+\beta\frac{f(z)g'(z)}{f'(z)g(z)}\right| \qquad (2)$$

are true for all $z \in U$*, then the function*

$$F_\gamma(z) = \left(\gamma\int_0^z u^{\gamma-1}f'(u)du\right)^{1/\gamma} \qquad (3)$$

is analytic and univalent in U*, where the principal branch is intended.*

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times I \longrightarrow C$, defined formally by

$$L(z,t) = \left(\gamma\int_0^{e^{-t}z} u^{\gamma-1}f'(u)du + \right. \qquad (4)$$

$$\left. + \frac{e^{(2-\gamma)t} - e^{-\gamma t}}{1+\alpha}z^\gamma\left[\frac{f'(e^{-t}z)}{g'(e^{-t}z)} + \beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\right]\right)^{1/\gamma}$$

is analytic in U_r , for all $t \in I$. Let us consider the function

$$h(z,t) = \frac{f'(e^{-t}z)}{g'(e^{-t}z)} + \beta\frac{f(e^{-t}z)}{g(e^{-t}z)} .$$

We have $h(0,t) = 1+\beta$ and we observe that $h(0,t) \ne 0$. Indeed, if $h(0,t) = 0$ then $\beta = -1$ and from the condition $|\alpha-\beta| \le |\beta+1|$ it follows $\alpha = -1$ which is a contradiction with the hypothesis $\alpha \ne -1$. Therefore there is a disk U_{r_1}, $0 < r_1 \le 1$, in which $h(z,t) \ne 0$ for all $t \in I$. Denoting

$$h_1(z,t) = \gamma\int_0^{e^{-t}z} u^{\gamma-1}f'(u)du$$

we have $h_1(z,t) = z^\gamma h_2(z,t)$ and is easy to see that h_2 is analytic in U_{r_1} for all $t \in I$, $h_2(0,t) = e^{-\gamma t}$. The function

$$h_3(z,t) = h_2(z,t) + \frac{e^{(2-\gamma)t} - e^{-\gamma t}}{1+\alpha}h(z,t)$$

is also analytic in U_{r_1} and

$$h_3(0,t) = \frac{e^{-\gamma t}}{1+\alpha}\left[\,(\alpha-\beta) + (1+\beta)e^{2t}\,\right].$$

Let us now prove that $h_3(0,t) \ne 0$ for any $t \in I$. We have $h_3(0,0) = 1$. Assume that there exists $t_0 > 0$ such that $h_3(0,t_0) = 0$. It follows $e^{2t_0} = (\beta-\alpha)/(1+\beta)$ and since $|\alpha-\beta| \le |\beta+1|$ we get $e^{2t_0} \le 1$ and this inequality is imposible. Therefore, there is a disk $U_{r_2},\ r_2 \in (0, r_1]$ in which $h_3(z,t) \neq 0$ for all $t \in I$. Then we can choose an uniform branch of $[h_3(z,t)]^{1/\gamma}$ analytic in U_{r_2}, denoted by $h_4(z,t)$, that is equal to

$$a_1(t) = e^{\frac{2-\gamma}{\gamma}t}\left[\frac{(\alpha-\beta)e^{-2t}+(1+\beta)}{1+\alpha}\right]^{1/\gamma}$$

at the origin, and for $a_1(t)$ we fix the principal branch, $a_1(0) = 1$. From these considerations, it follows that the relation (4) may be written as

$$L(z,t) = z \cdot h_4(z,t) = a_1(t)z + a_2(t)z^2 + \dots\ ,$$

and then the function $L(z,t)$ is analytic in U_{r_2}. Since $|\gamma - 1| < 1$ implies $\mathrm{Re}(2/\gamma) > 1$, it follows that $\lim_{t\to\infty} |a_1(t)| = \infty$. We saw also that $a_1(t) \neq 0$ for all $t \in I$.
From the analyticity of $L(z,t)$ in U_{r_2} it follows that there is a number $r_3,\ 0 < r_3 < r_2$, and a constant $K = K(r_3)$ such that

$$|\ L(z,t)/a_1(t)\ | \ <\ K, \qquad \forall z \in U_{r_3}, \quad t \in I,$$

In consequence, the family $\{\ L(z,t)/a_1(t)\ \}$ is normal in U_{r_3}. From the analyticity of $\partial L(z,t)/\partial t$, for all fixed numbers $T > 0$ and $r_4,\ 0 < r_4 < r_3$, there exists a constant $K_1 > 0$ (that depends on T and r_4) such that

$$\left|\ \frac{\partial L(z,t)}{\partial t}\ \right| \ <\ K_1, \qquad \forall z \in U_{r_4}, \quad t \in [0,T].$$

It follows that the function $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_4}. Let us set

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z}\Big/\frac{\partial L(z,t)}{\partial t} \tag{5}$$

and

$$w(z,t) = \frac{p(z,t)-1}{p(z,t)+1} \tag{6}$$

The function $p(z,t)$ is analytic in $U_r,\ 0 < r < r_4$. The function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, if the function $w(z,t)$ can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$and $t \in I$.
After computation we obtain:

$$w(z,t) = A(z,t)\cdot e^{-2t} + B(z,t)(1-e^{-2t})\ , \tag{7}$$

where

$$A(z,t) = \frac{\gamma(\alpha+1)g'(e^{-t}z) - 1 - \beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}}{1+\beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}} \tag{8}$$

$$B(z,t) = \gamma - 1 + \tag{9}$$

$$+\frac{\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)} - \frac{e^{-t}zg''(e^{-t}z)}{g'(e^{-t}z)} + \beta\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)}\left(1-\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}\right)}{1+\beta\frac{f(e^{-t}z)}{g(e^{-t}z)}\cdot\frac{g'(e^{-t}z)}{f'(e^{-t}z)}}\ .$$

From (1) and (2) we deduce that $w(z,t)$ is analytic in U. In view of (1), from (7) and (8) we have

$$|w(z,0)| = |A(z,0)| = \left| \frac{\gamma(\alpha+1)g'(z) - 1 - \beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)}}{1+\beta\frac{f(z)}{g(z)} \cdot \frac{g'(z)}{f'(z)}} \right| < 1 \tag{10}$$

For $z = 0,\ t > 0$, since $|\gamma(\alpha+1) - (1+\beta)| \le |\beta+1|$ and $|\gamma - 1| < 1$ we get

$$|w(0,t)| = \left| \frac{\gamma(\alpha+1) - (1+\beta)}{1+\beta} e^{-2t} + (\gamma-1)(1-e^{-2t}) \right| < 1 \tag{11}$$

If $t > 0$ is a fixed number and $z \in U$, $z \ne 0$, then the function $w(z,t)$ is analytic in $\bar{U}$ because $|e^{-t}z| \le e^{-t} < 1$ for all $z \in \bar{U}$ and it is known that

$$|w(z,t)| = \max_{|\zeta|=1} |w(\zeta,t)| = |w(e^{i\theta},t)|, \quad \theta = \theta(t) \in R \tag{12}$$

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and because $u \in U$, from (7), (8) and (9) taking into account (2) we get

$$|w(e^{i\theta},t)| \le 1. \tag{13}$$

From (10), (11), (12) and (13) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$. From Theorem 1.1 it results that $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t \in I$. But $L(z,0) = F_\gamma(z)$ and then the function F_γ defined by (3) is analytic and univalent in U.

Remark. Suitable choices of g yields various type of univalence criteria, so we can take $g(z) \equiv z,\ g(z) \equiv f(z)$ or $g(z) \equiv z \cdot f'(z)$.

If in Theorem 2.1 we take $g(z) \equiv z$ we have the following result.

Corollary 2.1. *Let $\alpha,\ \beta,\ \gamma$ be complex numbers such that $\alpha \ne -1$,*

$$|\alpha - \beta| \le |1+\beta|, \quad |\gamma - 1| < 1, \quad |\gamma(\alpha+1) - (\beta+1)| \le |\beta+1|\,,$$

and let $f \in A$. If the inequalities

$$\left| \gamma(\alpha+1) - 1 - \beta\frac{f(z)}{zf'(z)} \right| < \left| 1 + \beta\frac{f(z)}{zf'(z)} \right|$$

and

$$\left| \left[\gamma(\alpha+1) - 1 - \beta\frac{f(z)}{zf'(z)}\right] \cdot |z|^2 + \right.$$
$$\left. +(1-|z|^2)\left[(\gamma-1)\left(1+\beta\frac{f(z)}{zf'(z)}\right) + \frac{zf''(z)}{f'(z)} + \beta\left(1 - \frac{f(z)}{zf'(z)}\right)\right] \right| \le \left|1+\beta\frac{f(z)}{zf'(z)}\right|$$

are true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

Number of corollaries we can get for particular values of parameters α and β. We shall formulate only two: for $\beta = 0$ and for $\alpha = \beta = 0$.

For $\beta = 0$ we obtain from Corollary 2.1 a generalization of the well-known condition for univalence established by Ahlfors.

Corollary 2.2. *Let α, γ be complex numbers such that $\alpha \neq -1$, $|\alpha| \leq 1$, $|\gamma - 1| < 1$, $|\gamma(\alpha + 1) - 1| \leq 1$ and let $f \in A$. If the inequality*

$$\left| [\gamma(\alpha+1)-1]|z|^2 + (1-|z|^2)\left[\frac{zf''(z)}{f'(z)} + \gamma - 1\right]\right| \leq 1$$

is true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

In the case $\gamma = 1$ we get $F_1(z) = f(z)$ and we have the univalence criterion found by Ahlfors ([1]).

Corollary 2.3. ([1]). *Let $\alpha \in C$, $|\alpha| \leq 1$, $\alpha \neq -1$ and let $f \in A$. If the inequality*

$$\left| \alpha|z|^2 + (1-|z|^2)\frac{zf''(z)}{f'(z)} \right| \leq 1$$

holds for $z \in U$, then the function f is univalent in U.

For $\alpha = \beta = 0$ we get

Corollary 2.4. *Let $\gamma \in C$, $|\gamma - 1| < 1$. If the inequality*

$$\left| (1-|z|^2)\frac{zf''(z)}{f'(z)} + \gamma - 1 \right| \leq 1$$

is true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

We recognize here the expression $(1-|z|^2)zf''(z)/f'(z)$ which appears in the condition for univalence established by Becker. We know that if this value lies in the unit disk U, then the function f is univalent in U and we observe that if this value lies in a disk with the same radius 1, but with the center in the point $1-\gamma$, $|\gamma - 1| < 1$ we obtain the analyticity and the univalence of the function F_γ.

If in Theorem 2.1 we take $f \equiv g$ we have a very simple result given by

Corollary 2.5. *Let α, β, $\gamma \in C$ such that $\alpha \neq -1$, $|\alpha - \beta| \leq |\beta + 1|$, $|\gamma - 1| < 1$, $|\gamma(\alpha+1) - (\beta+1)| \leq |\beta + 1|$ and let $f \in A$. If the inequality*

$$\left| f'(z) - \frac{1+\beta}{\gamma(\alpha+1)} \right| < \frac{|1+\beta|}{|\gamma(\alpha+1)|}$$

is true for all $z \in U$, then the function F_γ defined by (3) is analytic and univalent in U.

Example. *Let $\gamma \in C$, $|\gamma - 1| < 1$. Then the function*

$$F(z) = z \cdot \left[1 + \frac{1-|\gamma|}{1+\gamma} \cdot z\right]^{1/\gamma}$$

is analytic and univalent in U.

To prove it consider the function $f \in A$ of the form

$$f(z) = z + \frac{1-|\gamma-1|}{2\gamma} \cdot z^2$$

and we apply corollary 2.5 with $\alpha = \beta$. So we have

$$\left| f'(z) - \frac{1}{\gamma} \right| = \left| \frac{\gamma - 1}{\gamma} + \frac{1 - |\gamma - 1|}{\gamma} \cdot z \right| \leq \frac{|\gamma - 1|}{|\gamma|} + \frac{1 - |\gamma - 1|}{|\gamma|} < \frac{1}{|\gamma|}.$$

Remark. For the case $\gamma = 1$ we have $F_1(z) = f(z)$ and from Theorem 2.1 we find the results obtained by S. Kanas and A. Lecko [3].

3. **Univalence criteria and geometric mean**

Substituting the arithmetic mean by the geometric one in the construction of the Löwner chain we obtain the following

Theorem 3.1. *Let α, β, γ be complex number such that $|\gamma - 1| < 1$, $Re\gamma > 1/2$ and let $f \in A$, $f'(z)f(z)/z \neq 0$ in U. If there exists a function $g \in A$, $g'(z)g(z)/z \neq 0$ in U, such that the inequalities*

$$\left| f'(z) \left(\frac{g'(z)}{f'(z)} \right)^{\alpha} \cdot \left(\frac{g(z)}{f(z)} \right)^{\beta} - 1 \right| < 1 \,, \tag{14}$$

$$\left| \left[f'(z) \cdot \left(\frac{g'(z)}{f'(z)} \right)^{\alpha} \left(\frac{g(z)}{f(z)} \right)^{\beta} - 1 \right] \cdot |z|^2 + \right. \tag{15}$$

$$\left. +(1 - |z|^2) \left[\alpha \left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)} \right) + \beta \left(\frac{zf'(z)}{f(z)} - \frac{zg'(z)}{g(z)} \right) + \gamma - 1 \right] \right| \leq 1$$

are true for all $z \in U$, then the function

$$F_\gamma(z) = \left(\gamma \int_0^z u^{\gamma - 1} f'(u) du \right)^{1/\gamma} \tag{16}$$

is analytic and univalent in U, where the principal branch is intended.

Proof. The method of the proof is similar to those of Theorem 2.1. Let us define

$$L(z,t) = \left[\int_0^{e^{-t}z} u^{\gamma - 1} f'(u) du + \right. \tag{17}$$

$$\left. + \left(e^{(2-\gamma)t} - e^{-\gamma t} \right) z^{\gamma} \left(\frac{f'(e^{-t}z)}{g'(e^{-t}z)} \right)^{\alpha} \left(\frac{f(e^{-t}z)}{g(e^{-t}z)} \right)^{\beta} \right]^{1/\gamma}$$

It can be shown that $L(z,t)$ is an analytic function in U_r, $r \in (0,1]$ for all $t \in I$, $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots$, where

$$a_1(t) = e^{\frac{2-\gamma}{\gamma} t} \left(1 + \frac{1 - \gamma}{\gamma} e^{-2t} \right)^{1/\gamma} \tag{18}$$

We fix a determination of $(1/\gamma)^{1/\gamma}$ denoted by δ. For $a_1(t)$ we fix the determination equal to δ for $t = 0$. Since $|\gamma - 1| < 1$ it follows that $\lim_{t \to \infty} |a_1(t)| = \infty$ and from $\mathrm{Re}\gamma > 1/2$ we have $|\gamma - 1| < |\gamma|$ and then $a_1(t) \neq 0$ for all $t \in I$.

Moreover, it can be prove that there is a disk U_{r_0}, $0 < r_0 < r$ such that $L(z,t)$ is locally absolutely continuous in I, locally uniform with respect to U_{r_0} and

$\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_0}. For the functions $p(z,t)$ and $w(z,t)$ defined in (5) and (6), by computation we get

$$w(z,t)=\left[\,f'(e^{-t}z)\left(\frac{g'(e^{-t}z)}{f'(e^{-t}z)}\right)^{\alpha}\left(\frac{g(e^{-t}z)}{f(e^{-t}z)}\right)^{\beta}-1\,\right]\cdot e^{-2t}+$$
$$+(1-e^{-2t})\left[\alpha\left(\frac{e^{-t}zf''(e^{-t}z)}{f'(e^{-t}z)}-\frac{e^{-t}zg''(e^{-t}z)}{g'(e^{-t}z)}\right)+\right.$$
$$\left.+\,\beta\left(\frac{e^{-t}zf'(e^{-t}z)}{f(e^{-t}z)}-\frac{e^{-t}zg'(e^{-t}z)}{g(e^{-t}z)}\right)+\gamma-1\right].$$

We observe that the function $w(z,t)$ is well-defined and analytic in U for each $t\in I$. The rest of the proof runs exactly as in Theorem 2.1. From Theorem 1.1 it results that the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, for each $t\in I$, in particular $L(z,0)$. But

$$L(z,0)=\left(\int_0^z u^{\gamma-1}f'(u)du\right)^{1/\gamma}$$

and then also the function F_γ defined by (16) is analytic and univalent in U.

For $g(z)\equiv z$ we can deduce the following

Corollary 3.1. *Let $\alpha,\ \beta,\ \gamma\in C$ such that $|\gamma-1|<1$, $Re\gamma>1/2$ and let $f\in A,\ f'(z)f(z)/z\neq 0$ in U. If the inequalities*

$$\left|\left(\frac{z}{f(z)}\right)^{\beta}(\,f'(z)\,)^{1-\alpha}-1\right|<1$$

$$\left|\left[\left(\frac{z}{f(z)}\right)^{\beta}(f'(z))^{1-\alpha}-1\right]|z|^2+\right.$$
$$\left.+(1-|z|^2)\left[\alpha\frac{zf''(z)}{f'(z)}+\beta\left(\frac{zf'(z)}{f(z)}-1\right)+\gamma-1\right]\right|\leq 1$$

hold for all $z\in U$, then the function F_γ defined by (16) is analytic and univalent in U.

For $\alpha=0$ and $\beta=1$, from Corollary 3.1 we get

Corollary 3.2. *Let $\gamma\in C,\ |\gamma-1|<1,\ Re\gamma>1/2$ and let $f\in A,\ f'(z)f(z)/z\neq 0$ in U. If the inequalities*

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1$$

$$\left|\left(\frac{zf'(z)}{f(z)}-1\right)|z|^2+(1-|z|^2)\left(\frac{zf'(z)}{f(z)}+\gamma-2\right)\right|\leq 1$$

hold for all $z\in U$, then the function F_γ defined by (16) is analytic and univalent in U.

For $g(z)\equiv f(z)$, from Theorem 3.1 we get the following useful result

Corollary 3.3. *Let $\gamma \in C$, $|\gamma - 1| < 1$, $Re\gamma > 1/2$ and let $f \in A$. If the inequality*

$$| f'(z) - 1 | < 1 \tag{19}$$

hold for all $z \in U$, then the function F_γ defined by (16) is analytic and univalent in U.

Indeed, the inequality (14) becomes (19) and the inequality (15) will be

$$\left| (f'(z) - 1)|z|^2 + (1 - |z|^2)(\gamma - 1) \right| \leq 1.$$

This inequality is true under the assumption $|\gamma - 1| < 1$ and in view of (19).

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