MODULES OVER TRIANGULATED CATEGORIES
AND LOCALIZATION

CIPRIAN MODOI

Abstract. For a compactly generated triangulated category we gives a new proof for the fact that the category of modules over its subcategory consisting of all compact objects it is not only the colocalization, but also the localization of the category of finitely presented modules over the full triangulated category. We do not only prove the existence of a right adjoint for the restriction functor, but we give it explicitly.

A problem arising in the study of (compactly generated) triangulated categories is to find some abelian categories closely related to a given triangulated one. A such category is the category of finitely presented contravariant functors defined on it with values in the category of abelian groups. We denote it here by Mod-$\mathcal{T}$, where $\mathcal{T}$ is the triangulated category. The Yoneda embedding gives an universal homological functor $\mathbf{h} : \mathcal{T} \to$ Mod-$\mathcal{T}$ [3, 5.1.18]. A result due to Neeman [3, 5.3.9] says, that a triangulated functor between two triangulated categories $\mathcal{T} \to \mathcal{S}$ have a right or a left adjoint if and only if the induced functor Mod-$\mathcal{T} \to$ Mod-$\mathcal{S}$ does. But it is also not easy to deal with the category Mod-$\mathcal{T}$, since it may be not well-powered [3, Appendix C], in the sense that an object may have a proper class (which is not a set) of subobjects (quotients). In the same work of Neeman [3], was observed that a "good" approximation of the category Mod-$\mathcal{T}$ is the category $\mathrm{Ex}((\mathcal{T}^\alpha)^{\mathrm{OP}}, \mathcal{A}b)$, whose objects are additive functors $(\mathcal{T}^\alpha)^{\mathrm{OP}} \to \mathcal{A}b$ which take coproducts fewer than α objects in products in $\mathcal{A}b$. Here α is a fixed regular cardinal, $\mathcal{T}^\alpha$ is the full subcategory of α-compact objects of $\mathcal{T}$, in the sense of the definition [3, 4.2.7], and it is supposed to be skeletally small. Precisely, the category $\mathrm{Ex}((\mathcal{T}^\alpha)^{\mathrm{OP}}, \mathcal{A}b)$ is the colocalization of Mod-$\mathcal{T}$ [3, 6.5.3]. In the case $\mathcal{C} = \mathcal{T}^{\aleph_0}$ that is $\alpha = \aleph_0$ we have $\mathrm{Ex}(\mathcal{C}^{\mathrm{OP}}, \mathcal{A}b) =$ Mod-$\mathcal{C}$ contains all functors $\mathcal{C}^{\mathrm{OP}} \to \mathcal{A}b$. In this note we find a new proof for the fact that Mod-$\mathcal{C}$ is not only the colocalization, but also the localization of Mod-$\mathcal{T}$, an explicit formula for the right adjoint of the restriction functor being also given.

A few words about terminology and notations: By $\mathcal{A}b$ we shall denote the category of abelian groups. We shall write $\mathcal{A} \to \mathcal{B}$ respectively $\mathcal{A}^{\mathrm{OP}} \to \mathcal{B}$ to emphasize that we deal with a covariant (contravariant) functor between two given categories $\mathcal{A}$ and $\mathcal{B}$. It is well–known that an associative ring R may be regarded as a preadditive

category with a single object, and then a right R-module means a functor $R^{\mathrm{op}} \to \mathcal{A}b$. The additive functors $\mathcal{A}^{\mathrm{op}} \to \mathcal{A}b$, defined on an arbitrary preadditive category $\mathcal{A}$ will also call (right) modules over $\mathcal{A}$, or simply $\mathcal{A}$-modules. We denote by $\mathcal{A}(a', a)$ and $\mathrm{Hom}_{\mathcal{A}}(M', M)$ the set of morphism between objects a' and a, in the category $\mathcal{A}$, respectively the class of all natural transformations between $\mathcal{A}$-modules M' and M.

For basic facts about abelian categories, we refer the reader to [4], and for the general theory of triangulated categories to [3]. Even if in the text all references concerning abelian categories are to [4], the personal experience of the author playing a rôle here, this things may be found also in many works, for example in Gabriel's [1].

By a (right) module over a preadditive category $\mathcal{T}$, we understand, as in the case of ordinary modules over a ring, an additive contravariant functor $M : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$. If $\mathcal{T}$ is skeletally small, then the class of all modules over $\mathcal{T}$ together with the natural transformations between them, form a Grothendieck category, denoted by Mod-$\mathcal{T}$ [4, chapter 4, 4.9], where the limits and the colimits are computed pointwise. Returning to the general case, a module N over the category $\mathcal{T}$ is called *finitely presented*, if there is an exact sequence of functors and natural transformations

$$\mathcal{T}(-, s) \to \mathcal{T}(-, t) \to N \to 0.$$

Denote by Mod-$\mathcal{T}$ the class of all finitely presented modules over $\mathcal{T}$. Note that, even if the class of all modules over $\mathcal{T}$ forms only a illegitimate category, in the sense that the class of the natural transformations between two such modules may be proper, this does not happen with Mod-$\mathcal{T}$. Indeed, by the Yoneda Lemma we infer that the class of all natural transformation between two finitely presented modules is actually a set, Mod-$\mathcal{T}$ together with natural transformations being a good defined category. If, in addition, $\mathcal{T}$ is triangulated, then by [3, 5.1.10], the category Mod-$\mathcal{T}$ is an abelian one.

Let $\mathcal{T}$ be a triangulated category, and $\mathcal{C}$ its full subcategory consisting of all compact objects. Recall that an object $c \in \mathcal{T}$ is called *compact* provided that the covariant functor $\mathcal{T}(c, -) : \mathcal{T} \to \mathcal{A}b$ commutes with direct sums. It is well–known, and also easy to see, that $\mathcal{C}$ is a *thick* subcategory of $\mathcal{T}$, that means, a triangulated subcategory which closed under direct summands. Throughout of this note we assume $\mathcal{T}$ has arbitrary coproducts, $\mathcal{C}$ is a skeletally small category, and it generates $\mathcal{T}$, i.e. $\mathcal{T}(c, x) = 0$ for all $c \in \mathcal{C}$ implies $x = 0$.

The functor $\mathbf{h} : \mathcal{T} \to $ Mod-$\mathcal{T}$, $\mathbf{h}(t) = \mathcal{T}(-, t)$ is a homological embedding, which send any object t of $\mathcal{T}$ into a projective object of Mod-$\mathcal{T}$. Moreover, since $\mathcal{T}$ is idemsplit (that is every idempotent $t \to t$ splits, for all $t \in \mathcal{T}$) [3, 1.6.8], every projective object of Mod-$\mathcal{T}$ is of this form [3, 5.1.11]. Restricting to $\mathcal{C}$ the images of $\mathbf{h}$ on each object $t \in \mathcal{T}$, we obtain a homological functor $\bar{\mathbf{h}} : \mathcal{T} \to $ Mod-$\mathcal{C}$, $\bar{\mathbf{h}}(t) = \mathcal{T}(-, t)|_{\mathcal{C}}$. Clearly $\bar{\mathbf{h}}$ commutes with coproducts, and for any $M \in $ Mod-$\mathcal{C}$, there is

an exact sequence
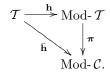
$$\bigoplus_{j\in J} \bar{\mathbf{h}}(d_j) \longrightarrow \bigoplus_{i\in I} \bar{\mathbf{h}}(c_i) \longrightarrow M \longrightarrow 0$$
$$\|\qquad\qquad\quad \|\qquad\qquad\quad \|$$
$$\bar{\mathbf{h}}\left(\bigoplus_{j\in J} d_j\right) \longrightarrow \bar{\mathbf{h}}\left(\bigoplus_{i\in I} c_i\right) \longrightarrow M \longrightarrow 0,$$

with d_j and c_i belonging to $\mathcal{C}$.

Since $\mathbf{h}$ is an universal homological functor [3, 5.1.18], it results an exact functor $\boldsymbol{\pi}$ making commutative the diagram

$$\mathcal{T} \xrightarrow{\ \mathbf{h}\ } \text{Mod-}\mathcal{T}$$
$$\underset{\bar{\mathbf{h}}}{\searrow} \quad \downarrow \boldsymbol{\pi}$$
$$\text{Mod-}\mathcal{C}.$$

Because every additive contravariant functor takes finite coproducts to products, we lie in the hypothesis of [3, chapter 6]. It follows that $\boldsymbol{\pi}(M) = M|_{\mathcal{C}}$ [3, 6.5.2], and Mod-$\mathcal{C}$ is the colocalization of Mod-$\mathcal{T}$, what means, the functor $\boldsymbol{\pi}$ has a fully–faithful left adjoint $\mathbf{L} : \text{Mod-}\mathcal{C} \to \text{Mod-}\mathcal{T}$. This adjoint is determined by its right exactness, and by the equality

$$\mathbf{L}\left(\bigoplus_{i\in I} \bar{\mathbf{h}}(c_i)\right) = \mathbf{h}\left(\bigoplus_{i\in I} c_i\right),$$

for all $c_i \in \mathcal{C}$ [3, 6.5.3]. Denote by $v : 1_{\text{Mod-}\mathcal{C}} \to \boldsymbol{\pi}\mathbf{L}$ and $u : \mathbf{L}\boldsymbol{\pi} \to 1_{\text{Mod-}\mathcal{T}}$ the unit, respectively the counit, of this adjunction. It is well–known that the fully–faithfulness of $\mathbf{L}$ is equivalent to the existence of an inverse for v [4, chapter 1, 13.11].

Lemma 1. *Any projective object P of* Mod-$\mathcal{C}$ *is of the form, $\bar{\mathbf{h}}(c)$ for an object $c = \bigoplus_{i\in I} c_i \in \mathcal{T}$, with $c_i \in \mathcal{C}$, and the induced map*

$$\mathcal{T}(c,x) \to \text{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \bar{\mathbf{h}}(x))$$

is an isomorphism for all $x \in \mathcal{T}$.

Proof. A projective object P of Mod-$\mathcal{C}$ is a direct summand of a direct sum $\bigoplus_{j\in J} \mathcal{C}(-, d_j)$, and because $\mathcal{C}$ is idemsplit, it follows $P \cong \bar{\mathbf{h}}(\bigoplus_{i\in I} c_i) = \bar{\mathbf{h}}(c)$.

Using the isomorphism of adjunction, and then the Yoneda isomorphism, we have

$$\text{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \bar{\mathbf{h}}(x)) = \text{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), \boldsymbol{\pi}(\mathbf{h}(x))) \cong \text{Hom}_{\mathcal{T}}(\mathbf{L}(\bar{\mathbf{h}}(c)), \mathbf{h}(c))$$
$$\cong \text{Hom}_{\mathcal{T}}(\mathbf{h}(c), \mathbf{h}(x)) \cong \mathcal{T}(c,x).$$

$\square$

We record also an analogous for injectives:

Lemma 2. [2, Lemma 1] *Any injective object Q of* Mod-$\mathcal{C}$ *is of the form* $\bar{\mathbf{h}}(q)$, *for an object* $q \in \mathcal{T}$, *and the induced map*

$$\mathcal{T}(x,q) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), \bar{\mathbf{h}}(q))$$

is an isomorphism for all $x \in \mathcal{T}$.

Lemma 3. *The assignement* $M \mapsto \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M)$ *gives a functor*

$$\mathbf{R} : \mathrm{Mod}\text{-}\mathcal{C} \to \mathrm{Mod}\text{-}\mathcal{T}.$$

Proof. The unique problem which arises is that $\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M) : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ is actually finitely presented, for all $M \in \mathrm{Mod}\text{-}\mathcal{C}$.

Choose an injective resolution for M:

$$0 \to M \to Q_1 \to Q_2.$$

Fix an object $x \in \mathcal{T}$. Applying the left exact functor $\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), -)$ to this injective resolution, and using Lemma 2, it follows that there are two objects $q_1, q_2 \in \mathcal{T}$, and a commutative diagram of abelian groups:

$$
\begin{array}{ccccc}
0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), M) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), Q_1) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), Q_2) \\
\| & & \| & & \| \\
0 \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(x), M) & \longrightarrow & \mathcal{T}(x, q_1) & \longrightarrow & \mathcal{T}(x, q_2).
\end{array}
$$

Therefore $\mathrm{Hom}(\bar{\mathbf{h}}(-), M)$ is pointwise the kernel of the natural transformation $\mathcal{T}(-, q_1) \to \mathcal{T}(-, q_2)$ between two finitely presented $\mathcal{T}$-modules. Then, by [3, 5.1.10], this functor belongs to Mod-$\mathcal{T}$. $\square$

Now we are ready to give the main result of this note.

Theorem 4. *The functor* $\mathbf{R} : \mathrm{Mod}\text{-}\mathcal{C} \to \mathrm{Mod}\text{-}\mathcal{T}$ *is the fully–faithful right adjoint of the functor* $\boldsymbol{\pi} : \mathrm{Mod}\text{-}\mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{C}$, *so the category* Mod-$\mathcal{C}$ *is not only the colocalization, but also the localization of the category* Mod-$\mathcal{T}$.

Proof. Let $c \in \mathcal{C}$, and $M : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ be a $\mathcal{C}$-module. Then, the Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), M) = \mathrm{Hom}_{\mathcal{C}}(\mathcal{T}(-, c)|_{\mathcal{C}}, M) \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}(-, c), M) \cong M(c)$$

shows that $\boldsymbol{\pi}\mathbf{R}(M) = \mathbf{R}(M)|_{\mathcal{C}} = \mathrm{Hom}(\bar{\mathbf{h}}(-), M)|_{\mathcal{C}}$ is naturally isomorphic to M. Denote by $v' : \boldsymbol{\pi}\mathbf{R} \to 1_{\mathrm{Mod}\text{-}\mathcal{C}}$ this isomorphism.

Let now $N : \mathcal{T}^{\mathrm{op}} \to \mathcal{A}b$ be a finitely presented $\mathcal{T}$-module. Then we have

$$\mathbf{R}\boldsymbol{\pi}(N) = \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), \boldsymbol{\pi}(N)) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}(\bar{\mathbf{h}}(-)), N)$$
$$= \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)), N),$$

and again an Yoneda isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(-), N) \cong N.$$

The counit $u_{\mathbf{h}(-)} : \mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)) \to \mathbf{h}(-)$ of the adjunction between $\mathbf{L}$ and $\boldsymbol{\pi}$ gives a morphism $\mathrm{Hom}_{\mathcal{T}}(u_{\mathbf{h}(-)}, N)$, which induced by the above isomorphisms an another one

$$u'_N : N \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{h}(-), N) \to \mathrm{Hom}_{\mathcal{T}}(\mathbf{L}\boldsymbol{\pi}(\mathbf{h}(-)), N) \cong \mathbf{R}\boldsymbol{\pi}(N),$$

so a natural transformation $u' : 1_{\mathrm{Mod}\text{-}\mathcal{T}} \to \mathbf{R}\boldsymbol{\pi}$.

Fix $c \in \mathcal{C}$, $t \in \mathcal{T}$, $M \in \mathrm{Mod}\text{-}\mathcal{C}$ and $N \in \mathrm{Mod}\text{-}\mathcal{T}$. The maps $\mathbf{R}(v'_M)$, $v'_{\boldsymbol{\pi}(N)}$ are clearly isomorphisms since v' is so. Moreover, the maps

$$(\boldsymbol{\pi}(u'_N))_c : \boldsymbol{\pi}(N) = N(c) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(c), N|_{\mathcal{C}})$$

and

$$(u'_{\mathbf{R}(M)})_t : \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(t), M) \to \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(t), \mathrm{Hom}_{\mathcal{C}}(\bar{\mathbf{h}}(-), M)|_{\mathcal{C}})$$

are isomorphisms too, by an analogous argument to the one used for v. Hence, the naturality of these morphisms implies the equalities

$$\mathbf{R}(v'_M) u'_{\mathbf{R}(M)} = 1_{\mathbf{R}(M)} \text{ and } v'_{\boldsymbol{\pi}(N)} \boldsymbol{\pi}(u'_N) = 1_{\boldsymbol{\pi}(N)},$$

which show that $\mathbf{R}$ is the right adjoint of $\boldsymbol{\pi}$, with the unit u' and the counit v'.

Finally the fully–faithfulness of $\mathbf{R}$ is equivalent, by [4, chapter 1, 13.10], to the fact that v' is invertible. $\qquad\square$

Remark 5. The subcategory $\mathrm{Ker}\,\boldsymbol{\pi}$ of $\mathrm{Mod}\text{-}\mathcal{T}$, consisting of the objects sent by $\boldsymbol{\pi}$ into 0 is both localizing and colocalizing, and the categories $\mathrm{Mod}\text{-}\mathcal{T}/\mathrm{Ker}\,\boldsymbol{\pi}$, $\mathrm{Mod}\text{-}\mathcal{C}$ and $\mathrm{Ker}\,\boldsymbol{\pi}\backslash\mathrm{Mod}\text{-}\mathcal{T}$ are all equivalent.

References

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"Babeş-Bolyai" University, 3400 Cluj-Napoca, Romania
E-mail address: cip7@math.ubbcluj.ro