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THE FIRST EIGENVALUE AND THE EXISTENCE RESULTS

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Abstract. In this paper we establish some conditions to existence for the solution of the boundary value problem

$$-\frac{1}{q(x)}\left(p(x)\,u'(x)\right)' = f\left(x, u(x), w(p,q)\,u'(x)\right), x \in (0,h)$$
$$u(0) = u(1) = 0$$

The hypotheses from the main result contain assumption on the first eigenvalue of some particular Sturm-Liouville problem. Using the lower boundary for the first eigenvalue, we can give some conditions of existence.

1. Introduction and notation

We consider the equation

$$-\left(p(x)\,u'(x)\right)' + q(x)\,u(x) = \lambda r(x)\,u(x) \tag{1}$$

for $x \in [0,h]$, where $p, p', q, r \in C(0,h)$ and satisfies $p(x) \geq p_0 > 0$, $q(x) \geq 0$, $r(x) \geq r_0 > 0$ for $x \in [0,h]$. The *Sturm-Liouville problem* is to find all *eigenvalues* λ for which the equation (1) has a nontrivial solution which satisfy the boundary condition

$$\alpha u(0) + \beta u'(0) = 0$$
$$\gamma u(h) + \delta u'(h) = 0,$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 \neq 0$ and $\gamma^2 + \delta^2 \neq 0$. The corresponding nontrivial solution is called an *eigenfunction*.

Example 1.1. For the problem

$$-u''(x) = \lambda u(x), x \in [0,\pi]$$
$$u(0) = (\pi) = 0$$

the eigenvalue are $\lambda_k = k^2$, $k \in \mathbb{N}$ and the corresponding eigenfunction is $u_k(x) = A_k \sin kx$, $k \in \mathbb{N}$.

In general, the first eigenvalue λ_1 of the Sturm-Liouville problem is too difficult to determinate. Using the Weinstein's method of intermediate problem we can find a lower boundary for λ_1 see [4]), and by Rayleigh-Ritz method it's possible to determinate an upper boundary for λ_1 .

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Example 1.2. [2] Let be the Sturm-Liouville problem

$$-\left(p\left(x\right)u'\left(x\right)\right)' + q\left(x\right)u\left(x\right) = \lambda u\left(x\right),\ x \in [0, h]$$

$$u\left(0\right) = u\left(h\right) = 0$$

where $p, p', q \in C\left(0, h\right)$, $0 < p_0 \leq p\left(x\right) \leq p_1$ and $0 \leq q\left(x\right) \leq q_1$ on $[0, h]$. We have the next approximation for the eigenvalues of this problem

$$\frac{p_0\pi^2k^2}{h^2} \leq \lambda_k \leq \frac{p_1\pi^2k^2}{h^2} + q_1, k \in \mathrm{N}$$

In the sequel, we make the following notation:

(N_1) R_β is the set of all measurable functions $q : (0, h) \to [0, \infty)$ such that

$$\int_0^h \left[q\left(x\right)\right]^\beta dx = 1$$

where β is a real number, $\beta \neq 0$;

(N_2) $m_\beta = \inf\limits_{q \in R_\beta} \lambda_1$ and $M_\beta = \sup\limits_{q \in R_\beta} \lambda_1$;

(N_3) R_α is the set of nonnegative measurable functions p on $(0, h)$ such that

$$\int_0^h \left[p\left(x\right)\right]^\alpha dx = 1$$

where α is a real number, $\alpha \neq 0$;

(N_4) $m_\alpha = \inf\limits_{q \in R_\alpha} \lambda_1$ and $M_\alpha = \sup\limits_{q \in R_\alpha} \lambda_1$;

(N_5) B is the Euler beta function $B\left(a, b\right) = \int_0^1 x^{a-1}\left(1 - x\right)^{b-1} dx$;

(N_6) $C\left(\alpha\right) = \begin{cases} \frac{2\alpha+1}{\alpha}\left(\frac{\alpha+1}{2\alpha+1}\right)^{1-\frac{1}{\alpha}} B^2\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2\alpha}\right), \text{ for } \alpha \in (-\infty, -1) \cup (0, +\infty) \\ -4\frac{2\alpha+1}{\alpha}\left(\frac{\alpha+1}{2\alpha+1}\right)^{1-\frac{1}{\alpha}}\left(\int_0^\infty \frac{dt}{(1+t^2)^{\frac{1}{2}-\frac{1}{2\alpha}}}\right), \text{ for } \alpha \in \left(-\frac{1}{2}, 0\right) \end{cases}$;

(N_7) The set

$$\Gamma = L_q^2\left(0, 1\right) =$$
$$\left\{u : [0, 1] \to \mathbb{R}; u \text{ is measurable function and } \int_0^1 q\left(x\right)\left|u\left(x\right)\right|^2 dx < \infty\right\}.$$

is endowed with the inner product

$$\left(u, v\right)_\Gamma = \int_0^1 q\left(x\right)u\left(x\right)v\left(x\right)dx \tag{2}$$

and the norm

$$\left\|u\right\|_\Gamma = \left(\int_0^1 q\left(x\right)\left|u\left(x\right)\right|^2 dx\right)^{\frac{1}{2}}. \tag{3}$$

(N_8) $L_q^2\left(0, 1; \mathbb{R}^2\right) = \left\{u : [0, 1] \to \mathbb{R}^2; \int_0^1 q\left(x\right)\left|u\left(x\right)\right|^2 dx < \infty\right\}$.

(N_9) The set $H = \left\{u \in L_q^2(0,1); u \text{ is absolute continuous and } u' \in L_p^2(0,1)\right\}$ is endowed with the inner product

$$(u,v)_H = \int_0^1 p(x)\, u'(x)\, v'(x)\, dx \tag{4}$$

and the norm

$$\|u\|_H = \left(\int_0^1 p(x)\left|u'(x)\right|^2 dx\right)^{\frac{1}{2}}. \tag{5}$$

Let us consider the Sturm-Liouville problem

$$u''(x) + \lambda q(x)\, u(x) = 0, x \in (0,h) \tag{6}$$
$$u(0) = u(h) = 0$$

The variational principle implies that the first eigenvalue λ_1 can be founds as

$$\lambda_1 = \inf_{\substack{u \in C_0^\infty(0,h) \\ u \neq 0}} \frac{\int_0^h \left[u'(x)\right]^2 dx}{\int_0^h q(x)\left[u(x)\right]^2 dx} \tag{7}$$

In the following, we remainder a result of Y. Egorov and V. Kondratiev

Lemma 1.1. [1] *If $\beta > 1$, then*

$$m_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(\beta-1)^{1+\frac{1}{\beta}}}{\beta(2\beta-1)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}\right) \text{ and } M_\beta = \infty.$$

If $\beta = 1$, then $M_1 = \infty$ and $m_1 = \frac{4}{h}$.
If $0 < \beta < \frac{1}{2}$, then

$$M_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(1-\beta)^{1+\frac{1}{\beta}}}{\beta(1-2\beta)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2\beta}\right) \text{ and } m_\beta = 0.$$

If $\beta < 0$, then

$$M_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(1-\beta)^{1+\frac{1}{\beta}}}{\beta(1-2\beta)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2\beta}\right) \text{ and } m_\beta = 0.$$

If $\frac{1}{2} \leq \beta < 1$, then $M_\beta = \infty$ and $m_\beta = 0$.

For the Sturm-Liouville problem

$$\left(p(x)\, u'(x)\right)' + \lambda u(x) = 0, \text{ for } x \in (0,1) \tag{8}$$
$$u(0) = u(1) = 0.$$

The first eigenvalue for this problem is given by

$$\lambda_1 = \inf_{\substack{u \in C_0^\infty(0,h) \\ u \neq 0}} \frac{\int_0^1 p(x)\left[u'(x)\right]^2 dx}{\int_0^1 \left[u(x)\right]^2 dx}. \tag{9}$$

We have the following result

Lemma 1.2. [1] *If* $\alpha > -\frac{1}{2}$, $\alpha \neq 0$ *then* $M_\alpha = C(\alpha)$ *and* $m_\alpha = 0$.
If $\alpha < -1$ *then* $m_\alpha = C(\alpha)$ *and* $M_\alpha = \infty$.
If $-1 \leq \alpha \leq -\frac{1}{2}$, *then* $M_\alpha = \infty$ *and* $m_\alpha = 0$.

2. Existence results

In that follows, we assume that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the *Caratheodory condition*, i.e.

(i) the application $f(x, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continuous a.e. for $x \in [0,1]$;
(ii) the application $f(\cdot, s) : [0,1] \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}^2$.

Let us consider the nonlinear boundary value problem

$$-\frac{1}{q(x)}\left(p(x)\, u'(x)\right)' = f\left(x, u(x), w(x)\, u'(x)\right), \text{ for } x \in (0,1) \tag{10}$$

$$u(0) = u(1) = 0.$$

Consider the operator $A : H \to \Gamma$ defined by

$$A(u)(x) = -\frac{1}{q(x)}\left(p(x)\, u'(x)\right)'. \tag{11}$$

We have

$$\begin{aligned}(Au, u)_\Gamma &= \int_0^1 q(x)\left[-\frac{\left(p(x)\, u'(x)\right)'}{q(x)}\right] u(x)\, dx \\ &= -p(x)\, u'(x)\, u(x)\big|_0^1 + \int_0^1 p(x)\left(u'(x)\right)^2 dx = \|u\|_H^2.\end{aligned}$$

Hence,

$$\|u\|_\Gamma^2 \leq \frac{1}{\lambda_1}(Au, u)_\Gamma \leq \frac{1}{\lambda_1}\|Au\|_\Gamma \cdot \|u\|_\Gamma.$$

Therefore,

$$\|u\|_\Gamma \leq \frac{1}{\lambda_1}\|Lu\|_\Gamma. \tag{12}$$

Theorem 2.1. *Suppose that*

(H_1) $w(x) \leq \sqrt{\frac{p(x)}{q(x)}}$ *on* $[0,1]$;
(H_2) *the application* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies the Caratheodory conditions and*

$$|f(x, s, t)| \leq a|t| + b|s| + g$$

for every $x \in (0,1)$; $t, s \in \mathbb{R}$ *and* $g \in \Gamma$;
(H_3) *there exist* $a, b \in [0, \infty)$ *small enough that*

$$\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}} < 1.$$

Then, the problem (10) has at least one solution in H.

Proof. For the beginning, we write problem (10) as a fixed point problem. For this, consider the operator $J : H \to L_q^2\left(0,1;\mathbb{R}^2\right)$ given by

$$J(u) = (u, u')$$

and the Nemitskii operator $N_f : L_q^2\left(0,1;\mathbb{R}^2\right) \to \Gamma$ defined by

$$N_f(u)(x) = f(x, u_1(x), w(x)u_2(x))$$

where $u = (u_1, u_2)$. The hypothesis (H_2) ensures that the Nemitskii operator is well defined and continuous, see [3] for details. We have the diagram

$$H \xrightarrow{J} L_q^2\left(0,1;\mathbb{R}^2\right) \xrightarrow{N_f} \Gamma \xrightarrow{A^{-1}} H$$

Now, we have that the operator $T : H \to H$, $T = A^{-1}N_f J$ is completely continuous and the problem (10) is equivalent to the equation

$$Tu = u,\ u \in H.$$

We have $\|T\|_H^2 = (AT, T)_\Gamma \le \|T\|_\Gamma \cdot \|AT\|_\Gamma = \|T\|_\Gamma \cdot \|n_f\|_\Gamma$. ¿From (12), we obtain $\|T\|_\Gamma \le \frac{1}{\lambda_1}\|AT\|_\Gamma = \frac{1}{\lambda_1}\|N_f\|_\Gamma$. So,

$$\|T\|_H \le \frac{1}{\sqrt{\lambda_1}}\|N_f\|_\Gamma. \tag{13}$$

By (H_2) we have

$$\begin{aligned}
\|N_f\|_\Gamma &= \left(\int_0^1 q(x)\left|f(x, u(x), w(x)u'(x))\right|^2 dx\right)^{\frac{1}{2}} \\
&\le \left(\int_0^1 q(x)\left\{g(x) + a|u(x)| + b|w(x)u'(x)|\right\}^2 dx\right)^{\frac{1}{2}} \\
&\le \|g\|_\Gamma + a\|u\|_\Gamma + b\left(\int_0^1 q(x)w^2(x)\left(u'(x)\right)^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

Now, hypothesis (H_1) implies that

$$\begin{aligned}
\|N_f\|_\Gamma &\le \|g\|_\Gamma + a\|u\|_\Gamma + b\left(\int_0^1 p(x)\left(u'(x)\right)^2 dx\right)^{\frac{1}{2}} \\
&\le \|g\|_\Gamma + a\|u\|_\Gamma + b\|u\|_H.
\end{aligned}$$

Since $\|u\|_\Gamma^2 \le \frac{1}{\lambda_1}\|u\|_H^2$, results $\|N_f\|_\Gamma \le \|g\|_\Gamma + \frac{a}{\sqrt{\lambda_1}}\|u\|_H + b\|u\|_H$. Hence, by (13), we obtain

$$\|Tu\|_H \le \frac{\|g\|_\Gamma}{\sqrt{\lambda_1}} + \left(\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}}\right)\|u\|_H.$$

Now, conform to hypothesis (H_3) we can find a real number $r > 0$ such that

$$\|Tu\|_H < \|u\|_H \text{ for } u \in H \text{ with } \|u\| \ge r.$$

By Lerray - Schauder principle, result that equation $Tu = u$ has at least one solution in H. □

In a similar way, we can prove the next result

Theorem 2.2. *Let us consider the boundary value problem*

$$-\frac{1}{q(x)}\left(p(x)u'(x)\right)' = f(x, u(x), u'(x)), \text{ for any } x \in (0,1) \tag{14}$$

$$u(0) = u(1) = 0$$

Suppose that the mapping f satisfies H_2 and

(H_4) *there exist $a, b \in (0, \infty)$ small enough that*

$$\frac{a}{\lambda_1} + \frac{b}{\sqrt{\lambda_1}}\sqrt{\frac{q_1}{p_0}} < 1$$

Then, the problem (14) has at least one solution in H.

An analogous result remains if we consider the interval $[0, h]$.

Example 2.3. For the Sturm - Liouville problem

$$-u'' = \lambda(1 + \sin x)u, \text{ for } u \in [0, \pi]$$

$$u(0) = u(\pi) = 0$$

it can establish the inequality $0.5394 \leq \lambda_1 \leq 0.54088$, see [4]. So, the boundary value problem

$$\frac{1}{1 + \sin x} \cdot u''(x) = f(x, u, u'), \text{ for } x \in [0, \pi]$$

$$u(0) = u(1) = 0$$

has at least one solution if the mapping $f : [0, \pi] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory condition and

$$|f(x, s, t)| \leq \frac{|t| + |s|}{4} + g(x)$$

for $x \in (0, \pi)$ and $g \in L^2_q(0, \pi)$.

3. The First Eigenvalue and the Existence Results

Now, in Theorem 2.2 we put the estimation from Lemma 1.2 and obtain the following result

Theorem 3.3. *Consider the nonlinear boundary value problem*

$$-\left(p(x)u'(x)\right)' = f(x, u(x), u'(x)), \text{ for } x \in (0,1) \tag{15}$$

$$u(0) = u(1) = 0$$

Suppose that f satisfies (H_2) and

(H_5) *the nonnegative measurable mapping $p : [0,1] \to \mathbb{R}$ is such that*

$$p_0 = \inf_{x\in(0,1)} p(x) > 0 \text{ and } \int_0^1 p(x)^\alpha \, dx = 1 \text{ for } \alpha \le -1;$$

(H_6) *there exist the numbers* $a, b \in (0,\infty)$ *small enough that*

$$\frac{a}{m_\alpha} + \frac{b}{\sqrt{p_0 m_\alpha}} < 1,$$

with

$$m_\alpha = \frac{2\alpha+1}{\alpha}\left(\frac{\alpha+1}{2\alpha+1}\right)^{1-\frac{1}{\beta}} B^2\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{2\alpha}\right).$$

Then, the problem (15) *has at least one solution in* H.

By Theorem 2.2 and Lemma 1.1, we obtain

Theorem 3.4. *Consider the nonlinear boundary value problem*

$$-\frac{1}{q(x)}u''(x) = f(x, u(x), u'(x)), \text{ for } x \in (0,h) \tag{16}$$

$$u(0) = u(h) = 0$$

Suppose that f *satisfies* (H_2) *and*

(H_7) *the nonnegative measurable mapping* $q : [0,h] \to \mathbb{R}$ *is such that*

$$q_1 = \sup_{x\in(0,h)} q(x) < \infty \text{ and } \int_0^h q(x)^\beta \, dx = 1 \text{ for } \beta > 1$$

(H_8) *there exists the numbers* $a, b \in (0,\infty)$ *small enough that*

$$\frac{a}{m_\alpha} + \frac{b}{\sqrt{\frac{q_1}{m_\beta}}} < 1,$$

with

$$M_\beta = \left(\frac{1}{h}\right)^{2-\frac{1}{\beta}} \frac{(1-\beta)^{1+\frac{1}{\beta}}}{\beta(1-2\beta)^{\frac{1}{\beta}}} B^2\left(\frac{1}{2}, \frac{1}{2}-\frac{1}{2\beta}\right).$$

Then, the problem (16) *has at least one solution.*

References

[1] Y. Egorov, V. Kondratiev, *On Spectral Theory of Elliptic Operators*, Birkhäuser-Verlag, Berlin, 1996, 153-206.
[2] S. G. Mihlin, *Linear Equation with Partial Derivatives* (in Romanian), Bucharest, 1983.
[3] R. Precup, *Nonlinear Integral Equations*, (in Romanian), Babeş-Bolyai Univ., Cluj-Napoca, Romania, 1993, 77-97 (2001).
[4] Al. Weinstein, *On the Sturm-Lioville Theory and the Eigenvalues of intermediate Problems*, Numerische Mathematik, **5**, 1963, 238-245.

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