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APPROXIMATION BY GENERALIZED BRASS OPERATORS

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Abstract. We establish direct and converse theorems for generalized Brass operators and for parameter dependent Brass - type operators, respectively.

1. Introduction

In the paper [8], D. D. Stancu has introduced and investigated a linear positive operator $L_{n,r} : C[0,1] \to C[0,1]$ defined by

$$(L_{n,r}f)(x) \ = \ \sum_{k=0}^{n-r} p_{n-r,k}(x) \ \left[\ (1-x) \ f\left(\frac{k}{n}\right) + x \ f\left(\frac{k+r}{n}\right) \ \right], \tag{1}$$

where $n > 2r \geq 4$ and $p_{n-r,k}(x) = \binom{n}{k} x^k(1-x)^{n-r-k}$, $k = \overline{0, n-r}$. The operator $L_{n,2}$ has been given earlier by H. Brass in [4]. Stancu has established the convergence of the sequence $(L_{n,r})_{n>2r}$, the representation of the remainder in the approximation formula by means of the second - order divided differences and the estimate of the order of approximation using the classical moduli of continuity, respectively.

In what follows we give direct and converse theorems for the operator given above. The converse results will be of Berens - Lorentz type [3] and of strong converse inequality of type B, in the terminology of [7].

Furthermore, let us consider a new, parameter dependent linear positive operator $L^{\alpha}_{n,r} : C[0,1] \to C[0,1]$ defined by
$(L^{\alpha}_{n,r}f)(x) \ =$

$$= \sum_{k=0}^{n-r} w_{n-r,k}(x,\alpha) \cdot \left[\frac{1 - x(n-r-k)\alpha}{1+(n-r)\alpha} \cdot f\left(\frac{k}{n}\right) \ + \ \frac{x+k\alpha}{1+(n-r)\alpha} \cdot f\left(\frac{k+r}{n}\right) \right], \tag{2}$$

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where $n > 2r$ and

$$w_{n-r,k} = \begin{pmatrix} n-r \\ k \end{pmatrix} \cdot \frac{\Pi_{i=0}^{k-1}(x+i\alpha)\,\Pi_{j=0}^{n-r-k-1}(1-x+j\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha)},$$

where $k = \overline{0, n-r}$ and $\alpha \geq 0$ is a parameter which may depend only on the natural number n. In the case $\alpha = 0$, $L^0_{n,r}$ is the generalized Brass operator defined by (1). Similarly to (1), we shall prove direct and converse theorems for (2).

In the next sections we will use the weighted $K-$ functional for $f \in C[0,1]$ defined by

$$K_{2,\phi}(f,\delta) = \inf\left\{ \|f-g\| + \delta\,\|\phi^2 g''\| \;:\; g \in W^2_\infty(\phi) \right\}, \quad \delta \geq 0.$$

Here $\phi : [0,1] \to \mathbf{R}$ is an admissible step - weight function of the Ditzian - Totik modulus [1, pp. 8 - 9], $\|\cdot\|$ is the supremum norm on $C[0,1]$ and $W^2_\infty(\phi)$ consists of all functions $g \in C[0,1]$ such that g is twice continuously differentiable and $\|\phi^2 g''\|$ is finite. It is well - known that $K_{2,\phi}(f,\delta)$ and $\omega^2_\phi(f,\sqrt{\delta})$ are equivalent [1, p. 11, Theorem 2.1.1], where

$$\omega^2_\phi(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x\pm h\phi(x)\in[0,1]} |\, f(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))\,|$$

is the Ditzian - Totik modulus of smoothness of second order.

2. Direct and converse theorems

Our direct result is

Theorem 1. *Let $(L_{n,r})_{n>2r}$ be defined as in (1), $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $\phi : [0,1] \to \mathbf{R}$ an admissible step -weight function of the Ditzian - Totik modulus with ϕ^2 concave. Then*

$$|(L_{n,r}f)(x) - f(x)| \leq 4\ K_{2,\phi}\left(f, \frac{n+r(r-1)}{n^2} \cdot \frac{\varphi(x)^2}{\phi(x)^2} \right)$$

holds true for $x \in [0,1]$ and $f \in C[0,1]$.

Proof. By [8, p. 214, Theorem 2.1] we have $L_{n,r}(t-x,x) = 0$ and

$$L_{n,r}((t-x)^2,x) = \frac{n+r(r-1)}{n^2} \cdot \varphi(x)^2$$

On the other hand, the operator $L_{n,r}$ is bounded as follows from

$$\begin{aligned} |(L_{n,r}f)(x)| &\leq \sum_{k=0}^{n-r} p_{n-r,k}(x) \cdot \left[(1-x) \left| f\left(\frac{k}{n}\right) \right| + x \left| f\left(\frac{k+r}{n}\right) \right| \right] \\ &\leq \|f\| \cdot \sum_{k=0}^{n-r} p_{n-r,k}(x) = \|f\| \qquad (3) \end{aligned}$$

Now we use [2, p. 398, Theorem 1], obtaining the assertion of the theorem.

Corollary 1. *Let $L_{n,r}$, φ and ϕ be given as in Theorem* 1. *Then*

$$|(L_{n,r}f)(x) - f(x)| \;\le\; C\;\omega_\phi^2\left(\, f, \frac{\sqrt{n+r(r-1)}}{n}\cdot\frac{\varphi(x)}{\phi(x)}\,\right)$$

for $x \in [0,1]$ and $f \in C[0,1]$, where the constant C depends only on φ and ϕ.

Proof. It is a direct consequence of Theorem 1 and the equivalence between $K_{2,\phi}\left(f, \frac{n+r(r-1)}{n^2}\cdot\frac{\varphi(x)^2}{\phi(x)^2}\right)$ and $\omega_\phi^2\left(f, \frac{\sqrt{n+r(r-1)}}{n}\cdot\frac{\varphi(x)}{\phi(x)}\right)$.

In order to prove the next theorems we need some Bernstein type inequalities.

Lemma 1. *Let $\phi : [0,1] \to \mathbf{R}$ be an admissible step - weight function of the Ditzian - Totik modulus with ϕ^2 concave, $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ and $n > 2r \ge 4$. Then for $f \in C[0,1]$*

$$\|\varphi^2(L_{n,r}f)''\| \;\le\; 4\,(n-r)\,\|f\| \tag{4}$$

and for smooth functions $g \in C^2[0,1]$

$$\|\varphi^2(L_{n,r}g)''\| \;\le\; C_1(r)\,\|\varphi^2 g''\|, \tag{5}$$

$$\|\phi^2(L_{n,r}g)''\| \;\le\; C_1(r)\,\|\phi^2 g''\|, \tag{6}$$

where $C_1(r) = 50r^2 + 34r + 17$.

Proof. Let

$$(L^1_{n,r}f)(x) \;=\; \sum_{k=0}^{n-r} p_{n-r,k}(x)\; f\left(\frac{k}{n}\right), \quad x \in [0,1]$$

and

$$(L^2_{n,r}f)(x) \;=\; \sum_{k=0}^{n-r} p_{n-r,k}(x)\; f\left(\frac{k+r}{n}\right), \quad x \in [0,1].$$

Then

$$(L_{n,r}f)(x) \;=\; (1-x)\cdot(L^1_{n,r}f)(x) \;+\; x\cdot(L^2_{n,r}f)(x), \tag{7}$$

$x \in [0,1]$. Furthermore, let $\lambda^i_{n-r,k} : C[0,1] \to \mathbf{R}$ $(i = \overline{1,2})$ be positive linear functionals defined by $\lambda^1_{n-r,k}(f) = f\left(\frac{k}{n}\right)$ and $\lambda^2_{n-r,k}(f) = f\left(\frac{k+r}{n}\right)$, where $k = \overline{0, n-r}$ and $f \in C[0,1]$. Then $\lambda^1_{n-r,k}(1) = \lambda^2_{n-r,k}(1) = 1$. Moreover, if Π_1 denotes the set of all algebraic polynomials of degree at most one then $L^i_{n,r}(\Pi_1) \subset \Pi_1$ for $i = \overline{1,2}$. Therefore, by [2, p. 414, Lemma 3] we obtain

$$\varphi(x)^2\;|(L^i_{n,r}f)''(x)| \;\le\; 2\,(n-r)\,\|f\| \tag{8}$$

for $x \in [0,1]$, $n > 2r$ and $i = \overline{1,2}$.

On the other hand, in view of (7) we have

$$(L_{n,r}f)''(x) = -2(L^1_{n,r}f)'(x) + 2(L^2_{n,r}f)'(x) + (1-x)(L^1_{n,r}f)''(x) + x(L^2_{n,r}f)''(x). \tag{9}$$

Using [6, p. 305, (2.1)] we obtain

$$(L^1_{n,r}f)'(x) \;=\; (n-r)\sum_{k=0}^{n-r-1}\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right]\cdot p_{n-r-1,k}(x)$$

and

$$(L^2_{n,r}f)'(x) \;=\; (n-r) \sum_{k=0}^{n-r-1} \left[f\left(\frac{k+r+1}{n}\right) - f\left(\frac{k+r}{n}\right) \right] \cdot p_{n-r-1,k}(x).$$

Hence

$$\varphi(x)^2 \; |(L^i_{n,r}f)'(x) \;\le\; \frac{1}{2}\,(n-r)\,\|f\|, \tag{10}$$

$i=\overline{1,2}$. Then, by (9), (8) and (10) we obtain

$$\varphi(x)^2 \; |(L_{n,r}f)''(x)| \le (n-r)\|f\| + (n-r)\|f\| + (1-x)\cdot 2(n-r)\|f\|$$

$$+x\cdot 2(n-r)|f\| = 4(n-r)\|f\|,$$

which implies (4).

Furthermore,

$$\lambda^1_{n-r,k}\left(\left(t-\frac{k}{n-r}\right)^2\right) = \left(\frac{k}{n}-\frac{k}{n-r}\right)^2 = r^2\cdot\left(\frac{k}{n(n-r)}\right)^2 \le r^2\cdot\left(\frac{1}{n}\right)^2$$

and

$$\lambda^2_{n-r,k}\left(\left(t-\frac{k}{n-r}\right)^2\right) \;=\; \left(\frac{k+r}{n}-\frac{k}{n-r}\right)^2 = \left[\left(\frac{k}{n}-\frac{k}{n-r}\right)+\left(\frac{r}{n}\right)\right]^2$$

$$\le\; 2\left[\left(\frac{k}{n}-\frac{k}{n-r}\right)^2\left(\frac{r}{n}\right)^2\right] \le (2r)^2\cdot\left(\frac{1}{n}\right)^2$$

for $n>2r$ and $k=\overline{0,n-r}$. Thus, in view of [2, p. 144, Lemma 3] we have for $g\in C^2[0,1]$:

$$\|\phi^2(L^i_{n,r}g)''\| \;\le\; C'(r)\;\|\phi^2 g''\|, \tag{11}$$

$i=\overline{1,2}$, where $C'(r)=48r^2+32r+8$. By (9), we have
$\phi(x)^2\cdot|(L_{n,r}g)''(x)| \;\le$

$$\begin{aligned}\le\;& 2\;\phi(x)^2\cdot|(L^2_{n,r}g)'(x)-(L^1_{n,r}g)'(x)| + \\ +\;& (1-x)\cdot\phi(x)^2\;|(L^1_{n,r}g)''(x)| + x\cdot\phi(x)^2\;|(L^2_{n,r}g)''(x)|\end{aligned} \tag{12}$$

Therefore, in view of (11), we have to estimate $\phi(x)^2\cdot|(L^2_{n,r}g)'(x)-(L^1_{n,r}g)'(x)|$. Using Taylor's formulas

$$g\left(\frac{k+1}{n}\right) = g(x)\left(\frac{k+1}{n}-x\right)\;g'(x)+\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)\;g''(u)\;du$$

and

$$g\left(\frac{k}{n}\right) = g(x)+\left(\frac{k}{n}-x\right)\;g'(x)+\int_x^{\frac{k}{n}}\left(\frac{k}{n}-u\right)\;g''(u)\;du,$$

we obtain

$$
\begin{aligned}
&(L^1_{n,r}g)'(x) \ = \\
&\quad = \ (n-r)\sum_{k=0}^{n-r-1}\left[\left(g\left(\frac{k+1}{n}\right)-g(x)\right)-\left(g\left(\frac{k}{n}\right)-g(x)\right)\right]\cdot p_{n-r-1,k}(x) \\
&\quad = \ (n-r)\ \Bigg\{g'(x)\sum_{k=0}^{n-r-1}\left(\frac{k+1}{n}-x\right)p_{n-r-1,k}(x)\ + \\
&\qquad + \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)\ g''(u)\ du\ - \\
&\qquad - g'(x)\sum_{k=0}^{n-r-1}\left(\frac{k}{n}-x\right)\ p_{n-r-1,k}(x)\ - \\
&\qquad - \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-u\right)\ g''(u)\ du\Bigg\}
\end{aligned}
$$

But, if

$$
(B_{n-r-1}f)(x) \ = \ \sum_{k=o}^{n-r-1} p_{n-r-1,k}(x)\cdot f\left(\frac{k}{n-r-1}\right), \quad f\in C[0,1]
$$

then $B_{n-r-1}(t-x,x)=0$ and therefore

$$
\sum_{k=0}^{n-r-1}\left(\frac{k+1}{n}-x\right)\cdot p_{n-r-1,k}(x) \ = \ \frac{1}{n}-\frac{r+1}{n}\cdot x
$$

and

$$
\sum_{k=0}^{n-r-1}\left(\frac{k}{n}-x\right)\cdot p_{n-r-1,k}(x) \ = \ -\frac{r+1}{n}\cdot x,
$$

respectively. Thus

$$
\begin{aligned}
&(L^1_{n,r}g)'(x) \ = \\
&\quad = \ (n-r)\cdot\Bigg\{\frac{1}{n}\cdot g'(x)+\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)\ g''(u)\ du\ - \\
&\quad - \ \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-u\right)\ g''(u)\ du\Bigg\} \qquad (13)
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
&(L^2_{n,r}g)'(x) \ = \\
&\quad = \ (n-r)\cdot\Bigg\{\frac{1}{n}\cdot g'(x)+\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+r+1}{n}}\left(\frac{k+r+1}{n}-u\right)\ g''(u)\ du \\
&\quad - \ \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+r}{n}}\left(\frac{k+r}{n}-u\right)\ g''(u)\ du\Bigg\} \qquad (14)
\end{aligned}
$$

Thus (13) and (14) imply
$(L^2_{n,r}g)'(x) - (L^1_{n,r}g)'(x) \ =$

$$= \ (n-r)\cdot\left\{\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+r+1}{n}}\left(\frac{k+r+1}{n}-u\right)g''(u)\,du - \right.$$
$$- \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+r}{n}}\left(\frac{k+r}{n}-u\right)g''(u)\,du -$$
$$- \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)g''(u)\,du +$$
$$\left. + \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-u\right)g''(u)\,du\right\} \tag{15}$$

So we have to estimate $|\int_x^t (t-u)\,g''(u)\,du|$. Because ϕ^2 is concave, using [2, p. 399, (5)] we obtain

$$\Big|\int_x^t (t-u)\,g''(u)\,du\Big| \le \Big|\int_x^t |t-u|\cdot|g''(u)|\,du\Big| \le \Big|\int_x^t \frac{|t-u|}{\phi(u)^2}\,du\Big|\cdot\|\phi^2 g''\|$$
$$\le \Big|\int_x^t \frac{|t-x|}{\phi(x)^2}\,du\Big|\cdot\|\phi^2 g''\| \le \frac{(t-x)^2}{\phi(x)^2}\cdot\|\phi^2 g''\|$$

Hence

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\cdot\Big|\int_x^{\frac{k+r+1}{n}}\left(\frac{k+r+1}{n}-u\right)g''(u)\,du\Big| \le$$
$$\le \frac{\|\phi^2 g''\|}{\phi(x)^2}\cdot\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\left(\frac{k+r+1}{n}-x\right)^2,$$

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\cdot\Big|\int_x^{\frac{k+r}{n}}\left(\frac{k+r}{n}-u\right)g''(u)\,du\Big| \le$$
$$\le \frac{\|\phi^2 g''\|}{\phi(x)^2}\cdot\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\left(\frac{k+r}{n}-x\right)^2,$$

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\cdot\Big|\int_x^{\frac{k+1}{n}}\left(\frac{k+1}{n}-u\right)g''(u)\,du\Big| \le$$
$$\le \frac{\|\phi^2 g''\|}{\phi(x)^2}\cdot\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\left(\frac{k+1}{n}-x\right)^2$$

and

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \quad \cdot \quad \Big| \int_x^{\frac{k}{n}} \left(\frac{k}{n} - u\right) \; g''(u) \; du \, \Big| \le$$

$$\le \quad \frac{\|\phi^2 g''\|}{\phi(x)^2} \cdot \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left(\frac{k}{n} - x\right)^2,$$

respectively. Using again $B_{n-r-1}(t-x,x) = 0$ and $B_{n-r-1}(t^2,x) = x^2 + \frac{x(1-x)}{n-r-1}$ we obtain

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left(\frac{k+r+1}{n} - x\right)^2 \; =$$

$$\begin{aligned}
&= \sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left[\left(\frac{k}{n}\right) + 2\frac{k}{n} \cdot \left(\frac{r+1}{n} - x\right) + \left(\frac{r+1}{n} - x\right)^2\right] \\
&= \left(\frac{n-r-1}{n}\right)^2 \cdot \left[x^2 + \frac{x(1-x)}{n-r-1}\right] + 2 \cdot \frac{n-r-1}{n} \cdot \left(\frac{r+1}{n} - x\right) \cdot x \; + \\
&+ \left(\frac{r+1}{n} - x\right)^2 \\
&= \left(\frac{r+1}{n}\right)^2 \cdot x^2 - 2\left(\frac{r+1}{n}\right)^2 \cdot x + \left(\frac{r+1}{n}\right)^2 + \left(\frac{n-r-1}{n}\right)^2 \cdot \frac{x(1-x)}{n-r-1} \\
&\le \left(\frac{r+1}{n}\right)^2 \cdot (1-x)^2 + \frac{1}{4(n-r-1)} \; \le \; \left(\frac{r+1}{n}\right)^2 + \frac{1}{4} \cdot \frac{1}{n-r-1} \\
&= \frac{1}{n} \cdot \left[\frac{(r+1)^2}{n} + \frac{1}{4} \cdot \frac{n}{n-r-1}\right] \; \le \; \frac{1}{n} \cdot \left[\frac{1}{4} \cdot (r+1)^2 + 1\right], \qquad (16)
\end{aligned}$$

because

$$\sup \left\{\frac{n}{n-r-1} \; : \; n > 2r\right\} \; < \; \frac{2r}{2r-r-1} \; \le \; 4,$$

where $n > 2r \ge 4$. With similar arguments we obtain

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \cdot \left(\frac{k+r}{n} - x\right)^2 \; =$$

$$\begin{aligned}
&= \left(\frac{r+1}{n} \cdot x - \frac{r}{n}\right)^2 + \left(\frac{n-r-1}{n}\right)^2 \cdot \frac{x(1-x)}{n-r-1} \\
&\le \left(\frac{r}{n}\right)^2 + \frac{1}{4} \cdot \frac{1}{n-r-1} \; \le \; \frac{1}{n} \cdot \left(\frac{1}{4} \cdot r^2 + 1\right), \qquad (17)
\end{aligned}$$

$$\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x) \; \left(\frac{k+1}{n} - x\right)^2 \; =$$

$$
\begin{aligned}
&= \left(\frac{r+1}{n}\cdot x-\frac{1}{n}\right)^2+\left(\frac{n-r-1}{n}\right)^2\cdot\frac{x(1-x)}{n-r-1}\\
&\le \left(\frac{r}{n}\right)^2+\frac{1}{4}\cdot\frac{1}{n-r-1}\ \le\ \frac{1}{n}\cdot\left(\frac{1}{4}\cdot r^2+1\right) \qquad (18)
\end{aligned}
$$

and

$$
\sum_{k=0}^{n-r-1} p_{n-r-1,k}(x)\ \left(\frac{k}{n}-x\right)^2 =
$$

$$
\begin{aligned}
&= \left(\frac{r+1}{n}\right)^2\cdot x^2+\left(\frac{n-r-1}{n}\right)^2\cdot\frac{x(1-x)}{n-r-1}\\
&\le \left(\frac{r+1}{n}\right)^2+\frac{1}{4}\cdot\frac{1}{n-r-1}\ \le\ \frac{1}{n}\cdot\left[\frac{1}{4}(r+1)^2+1\right]. \qquad (19)
\end{aligned}
$$

Now, in view of (15), (16), (17), (18) and (19) we obtain
$2\ \phi(x)^2\cdot|(L^2_{n,r}g)'(x)-(L^1_{n,r}g)'(x)|\ \le$

$$
\begin{aligned}
&\le\ 2\ \frac{n-r}{n}\cdot\left\{\frac{1}{2}(r+1)^2+2+\frac{1}{2}r^2+2\right\}\cdot\|\phi^2g''\|\\
&\le\ \left(2r^2+2r+9\right)\cdot\|\phi^2g''\|.
\end{aligned}
$$

Hence, by (12) and (11) we get
$\phi(x)^2\ |(L_{n,r}g)''(x)|\ \le$

$$
\begin{aligned}
&\le\ (2r^2+2r+9)\cdot\|\phi^2g''\|+(1-x)\cdot C'(r)\ \|\phi^2g''\|+x\cdot C'(r)\ \|\phi^2g''\|\\
&=\ (50r^2+34r+17)\cdot\|\phi^2g''\|.
\end{aligned}
$$

This means that $\|\phi^2(L_{n,r}g)''\|\le C_1(r)\cdot\|\phi^2g''\|$, which was to be proved at (6).
If $\phi\equiv\varphi$ then we obtain (5), which completes the proof of lemma.

Remark 1. *If $\phi\equiv\varphi$ then, by Corollary 2, we have*

$$
\|L_{n,r}f-f\|\ \le\ C\ \omega^2_{\varphi}\left(f,\frac{\sqrt{n+r(r-1)}}{n}\right). \qquad (20)
$$

Thus our first converse theorem will constitute an inverse of (20). More precisely we have

Theorem 2. *If $f\in C[0,1]$ and $k>2r$, $n>2r$, $r\ge 2$ then we have*

$$
K_{2,\varphi}\left(f,\frac{n+r(r-1)}{n^2}\right)\ \le\ \|L_{k,r}f-f\|+C\cdot\frac{k}{n}\cdot K_{2,\varphi}\left(f,\frac{k+r(r-1)}{k^2}\right),
$$

where the constant C depends only on r (it can be chosen as $(r+1)C_1(r)$).

Proof. By Lemma 3 : (4) – (5) we obtain

$$
\begin{aligned}
&K_{2,\varphi}\left(f, \tfrac{n+r(r-1)}{n^2}\right) \;\le \\
&\le \quad \|f - L_{k,r}f\| + \frac{n+r(r-1)}{n^2}\cdot\|\varphi^2(L_{k,r}f)''\| \\
&\le \quad \|f - L_{k,r}f\| + \frac{n+r(r-1)}{n^2}\cdot\left\{\varphi^2(L_{k,r}(f-g))''\| + \|\varphi^2(L_{k,r}g)''\|\right\} \\
&\le \quad \|f - L_{k,r}f\| + \frac{n+r(r-1)}{n^2}\cdot\left\{4(k-r)\|f-g\| + C_1(r)\cdot\|\varphi^2 g''\|\right\} \\
&= \quad \|f - L_{k,r}f\| + \frac{n+r(r-1)}{n}\cdot\frac{k-r}{n}\cdot\left\{4\|f-g\| + C_1(r)\cdot\frac{1}{k-r}\cdot\|\varphi^2 g''\|\right\} \\
&\le \quad \|f - L_{k,r}f\| + \frac{r+1}{2}\cdot\frac{k}{n}\cdot\left\{4\|f-g\| + C_1(r)\cdot 2\cdot\frac{k+r(r-1)}{k^2}\cdot\|\varphi^2 g''\|\right\} \\
&\le \quad \|L_{k,r}f - f\| + C\cdot\frac{k}{n}\cdot\left\{\|f-g\| + \frac{k+r(r-1)}{k^2}\cdot\|\varphi^2 g''\|\right\}.
\end{aligned}
$$

Now taking infimum over all $g \in C^2[0,1]$ we obtain the assertion of our theorem.

Remark 2. *By Corollary 2, the implication*

$$
\omega_\phi^2(f,\delta) \;=\; O(\delta^\alpha) \;\Rightarrow\; |(L_{n,r}f)(x) - f(x)| \;\le\; C\;\left(\frac{\sqrt{n+r(r-1)}}{n}\cdot\frac{\varphi(x)}{\phi(x)}\right)^\alpha
$$

holds true for $\alpha \in (0,2)$.

The converse result of Berens - Lorentz type is included in the next theorem

Theorem 3. *Let* $(L_{n,r})_{n>2r}$ *be defined by (1),* $\varphi(x)\sqrt{x(1-x)}$, $x \in [0,1]$ *and* $\phi : [0,1] \to \mathbf{R}$ *an admissible step - weight function of the Ditzian - Totik modulus with* ϕ^2 *and* φ^2/ϕ^2 *concave functions on* $[0,1]$. *Then for* $f \in C[0,1]$ *and* $\alpha \in (0,2)$ *the pointwise approximation*

$$
|(L_{n,r}f)(x) - f(x)| \;\le\; C\;\left(\frac{\sqrt{n+r(r-1)}}{n}\cdot\frac{\varphi(x)}{\phi(x)}\right)^\alpha,
$$

$x \in [0,1]$ *implies* $\omega_\phi^2(f,\delta) \;\le\; C\;\delta^\alpha$, $\delta > 0$.

Proof. We mention that $C > 0$ denotes a constant in this theorem which may depends only on r and it can be different at each occurrence.

The statement of the theorem results from [2, p. 410, Theorem 3] with slight modification using Lemma 3. Indeed, because $n > 2r \ge 4$ we hqve $\frac{n+r(r-1)}{n} < \frac{r+1}{n}$. Thus

$$
|(L_{n,r}f)(x) - f(x)| \;\le\; C\;\left(\frac{r+1}{2}\right)^{\alpha/2}\cdot\left(n^{-1/2}\cdot\frac{\varphi(x)}{\phi(x)}\right)^\alpha.
$$

By Lemma 3 : (4) we have $|\varphi^2(L_{n,r}f)''\| \le 4n\|f\|$ for $f \in C[0,1]$. Using (6) and step by step the proof of [2, p. 410, Theorem 3] we obtain

$$
\omega_\phi^2(f,t) \;\le\; C\;\left(\delta^\alpha + \frac{t^2}{\delta^2}\cdot\omega_\phi^2(f,\delta)\right), \quad 0 < t \le \delta
$$

which yields the assertion of the theorem by the well - known Berens - Lorentz lemma [3].

To prove the strong converse inequality of type B for $L_{n,r}$ we need another lemmas.

Lemma 2. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$ *and* $n > 2r \geq 4$. *Then for* $f \in C[0,1]$

$$\|\varphi^3 (L_{n,r} f)'''\| \ \leq \ C_2 \ n^{3/2} \|f\| \tag{21}$$

and for smooth functions $g \in C^2[0,1]$

$$\|\varphi^3 (L_{n,r} g)'''\| \ \leq \ C_3(r) n^{1/2} \|\varphi^2 g''\|, \tag{22}$$

where $C_2 = \sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 11$ *and* $C_3(r) = 3C'(r) + 3\sqrt{2} = 144r^2 + 96r + 24 + 3\sqrt{2}$.

Proof. By (9) we have

$$(L_{n,r} f)'''(x) = -3(L^1_{n,r} f)''(x) + 3(L^2_{n,r} f)''(x) + (1-x)(L^1_{n,r} f)'''(x) + x(L^2_{n,r} f)'''(x).$$

Then

$$\begin{aligned} \varphi(x)^3 \cdot |(L_{n,r} f)'''(x)| \ &\leq \ 3\varphi(x)^3 |(L^1_{n,r} f)''(x)| + 3\varphi(x)^3 |(L^2_{n,r} f)''(x)| \\ &+ \ (1-x)\varphi(x)^3 |(L^1_{n,r} f)'''(x)| + x\varphi(x)^3 |(L^2_{n,r} f)'''(x)| \end{aligned} \tag{23}$$

Using (8) we obtain

$$\varphi(x)^3 |(L^i_{n,r} f)''(x)| \leq 2(n-r)\varphi(x)\|f\| \leq (n-r)\|f\| \tag{24}$$

for $x \in [0,1]$, $n > 2r$ and $i = \overline{1,2}$.

Furthermore, by means of the expressions

$$T_{n,s}(x) = \sum_{k=0}^{n} (k - nx)^s \ p_{n,k}(x), \quad n = 1, 2, \ldots, \quad s = 0, 1, 2, \ldots$$

we have the following estimates (see [6, pp. 303 - 304] and [7, p.128, Lemma 9.4.4]]) : $T_{n,2}(x) = n\varphi(x)^2$, $T_{n,4}(x) \leq 11n^2\varphi(x)^4$ and $T_{n,6}(x) \leq 61n^3\varphi(x)^6$, where $x \in [1/n, 1-1/n]$ and $n \geq 2$. In this case $\varphi(x) \geq \frac{1}{\sqrt{2n}}$, $x \in [1/n, 1-1/n]$. Then, for the Bernstein polynomials

$$(B_n f)(x) \ = \ \sum_{k=0}^{n} p_{n,k}(x) \ f\left(\frac{k}{n}\right), \quad f \in C[0,1]$$

and for $x \in [1/n, 1-1/n]$ we have

$$\varphi(x)^3 \cdot |(B_n f)'''(x)| \ =$$

$$= \frac{1}{\varphi(x)^3} \cdot \Big| \sum_{k=0}^{n} f\left(\frac{k}{n}\right)(k-nx)^3 p_{n,k}(x) - 3(1-2x)\sum_{k=0}^{n} f\left(\frac{k}{n}\right)(k-nx)^2 p_{n,k}(x) -$$

$$-(3nx(1-x) - 2x(1-x) + 1) \sum_{k=0}^{n} f\left(\frac{k}{n}\right)(k-nx)p_{n,k}(x) + 2nx(1-x)(1-2x) \Big|$$

$$\leq \frac{\|f\|}{\varphi(x)^3} \cdot \Big\{ (T_{n,6}(x))^{1/2} + 3|1-2x| \, (T_{n,4}(x))^{1/2}$$

$$+ |3n\varphi(x)^2 - 2\varphi(x)^2 + 1| \left(T_{n,2}(x)\right)^{1/2} + 2n|1-2x| \cdot \varphi(x)^2 \Big\}$$

$$\leq \frac{\|f\|}{\varphi(x)^3} \cdot \Big\{ \sqrt{61} n^{3/2} \varphi(x)^3 + 3\sqrt{11} n \varphi(x)^2 (3n\varphi(x)^2 + 1) n^{1/2} \varphi(x) + 2n\varphi(x)^2 \Big\}$$

$$\leq \|f\| \cdot \Big\{ \sqrt{61} n^{3/2} + 3\sqrt{22} n^{3/2} + 5n^{3/2} + 2\sqrt{2} n^{3/2} \Big\}$$

$$= \left(\sqrt{61} + 3\sqrt{22} + 5 + 2\sqrt{2}\right) n^{3/2} \|f\|. \tag{25}$$

On the other hand, by [1, p. 125, (9.4.3)] we have for $x \in [0, 1/n] \cup [1-1/n, 1]$ and $f \in C[0,1]$:

$$\varphi(x)^3 |(B_n f)'''(x)| \leq n^{-3/2} \cdot \left| n(n-1)(n-2) \sum_{k=0}^{n-3} \left[f\left(\frac{k+3}{n}\right) - 3f\left(\frac{k+2}{n}\right) + \right.\right.$$

$$\left.\left. + \; 3f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] p_{n-3,k}(x) \right| \leq 8n^{3/2} \|f\|. \tag{26}$$

Therefore, in view of (25) and (26) we get

$$\varphi(x)^3 |(B_n f)'''(x)| \leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|f\| \tag{27}$$

for $f \in C[0,1]$ and $x \in [0,1]$.

Moreover, $(L^1_{n,r} f)(x) = (B_{n-r} g^1_n)(x)$ and $(L^2_{n,r} f)(x) = (B_{n-r} g^2_n)(x)$, where $g^1_n(x) = f\left(\frac{n-r}{n} \cdot x\right)$, $x \in [0,1]$ and $g^2_n(x) = f\left(\frac{n-r}{n} \cdot x + \frac{r}{n}\right)$, $x \in [0,1]$, respectively. Then, by (27) we obtain

$$\begin{aligned} \varphi(x)^3 |(L^1_{n,r} f)'''(x)| &\leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|g^1_n\| \\ &\leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|f\| \end{aligned}$$

and

$$\begin{aligned} \varphi(x)^3 |(L^1_{n,r} f)'''(x)| &\leq (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|g^2_n\| \\ &\leq (\sqrt{61} + 3\sqrt{22} + 3\sqrt{2} + 5) n^{3/2} \|f\|. \end{aligned}$$

Hence, by (23) and (24) we have

$$\begin{aligned} \varphi(x)^3 |(L_{n,r} f)'''(x)| &\leq 6n\|f\| + (\sqrt{61} + 3\sqrt{22} + 2\sqrt{2} + 5) n^{3/2} \|f\| \\ &\leq C_2 n^{3/2} \|f\|, \end{aligned}$$

which was to be proved.

For (22) we use [7, p. 87, Lemma 8.4] :

$$\|\varphi^3 (B_n g)'''\| \leq \frac{3}{\sqrt{2}} n^{1/2} \|\varphi^2 g''\|.$$

Hence, by (23), replacing f by g, and (11) with $\phi \equiv \varphi$ we obtain

$$\varphi(x)^3 |(L_{n,r} g)'''(x)| \leq$$

$$\leq 3C'(r) \|\varphi^2 g''\| + (1-x) \cdot \varphi(x)^3 |(B_{n-r} g^1_n)'''(x)| + x \cdot \varphi(x)^3 |(B_{n-r} g^2_n)'''(x)|$$

$$\leq 3C'(r) \|\varphi^2 g''\| + (1-x) \cdot \frac{3}{\sqrt{2}} (n-r)^{1/2} \cdot \|\varphi^2 (g^1_n)''\| + x \cdot \frac{3}{\sqrt{2}} (n-r)^{1/2} \cdot \|\varphi^2 (g^2_n)''\|$$

$$\leq 3C'(r)\|\varphi^2 g''\| + (1-x)\cdot\frac{3}{\sqrt{2}}(n-r)^{1/2}\cdot\left(\frac{n-r}{n}\right)^2\cdot\|\varphi^2 g''\| +$$

$$+x\cdot\frac{3}{\sqrt{2}}(n-r)^{1/2}\cdot\left(\frac{n-r}{n}\right)^2\cdot\|\varphi^2 g''\| \ \leq\ (3C'(r)+3\sqrt{2})n^{1/2}\|\varphi^2 g''\|.$$

Hence $\|\varphi^3(L_{n,r}g)'''\| \ \leq\ C_3(r)n^{1/2}\|\varphi^2 g''\|$, which completes the proof of the lemma.

Lemma 3. *Let* $(L_{n,r})_{n>2r}$ *be defined by (1),* $\varphi(x) = \sqrt{x(1-x)}$, $x\in[0,1]$ *and* $a>0$, $E_{a,n} = \left\{x_0\in[0,1] \mid x_0 \pm an^{-1/2}\varphi(x_0)\in[0,1]\right\}$,

$$g_{M,n,x_0}(t) = \begin{cases} (t-x_0)^2, & \text{if} \quad |t-x_0|\geq Mn^{-1/2}\varphi(x_0) \\ 0, & \text{otherwise.}\end{cases}$$

Then $(L_{n,r}g_{M,n,x_0})(x_0)/(n^{-1}\varphi(x_0)^2) \ \to\ 0$ *as* $M\to\infty$ *uniformly in* n *and* $x_0\in E_{a,n}$.

Proof. Simple computations show, if $x_0\in E_{a,n}$ then $x_0\in\left[\frac{a^2}{n+a^2}, 1-\frac{a^2}{n+a^2}\right]$. This means that

$$\sqrt{n}\varphi(x_0) \ \geq\ \frac{a}{1+a^2}. \tag{28}$$

Therefore, by (7) we obtain

$$\frac{n}{\varphi(x_0)^2}\cdot\left(L_{n,r}g_{m,n,x_0}\right)(x_0) \ =$$

$$= \frac{n}{\varphi(x_0)^2}\cdot\left\{(1-x_0)\sum_{\left|\frac{k}{n}-x_0\right|\geq Mn^{-1/2}\varphi(x_0)} p_{n-r,k}(x_0)\left(\frac{k}{n}-x_0\right)^2 +\right.$$

$$\left. + x_0\sum_{\left|\frac{k+r}{n}-x_0\right|\geq Mn^{-1/2}\varphi(x_0)} p_{n-r,k}(x_0)\left(\frac{k+r}{n}-x_0\right)^2\right\}$$

$$\leq \frac{n}{\varphi(x_0)^2}\cdot\left\{(1-x_0)\sum_{k=0}^{n-r}\frac{1}{M^2}\cdot\frac{n}{\varphi(x_0)^2}\cdot p_{n-r,k}(x_0)\left(\frac{k}{n}-x_0\right)^4 +\right.$$

$$\left. + x_0\sum_{k=0}^{n-r}\frac{1}{M^2}\cdot\frac{n}{\varphi(x_0)^2}\cdot p_{n-r,k}(x_0)\left(\frac{k+r}{n}-x_0\right)^4\right\}$$

$$= \frac{1}{M^2}\cdot\left(\frac{n}{\varphi(x_0)^2}\right)^2\cdot\left\{(1-x_0)\left[\frac{1}{n^4}\cdot T_{n-r,4}(x_0) - 4\cdot\frac{rx_0}{n^4}\cdot T_{n-r,3}(x_0) +\right.\right.$$

$$\left. + 6\cdot\frac{(rx_0)^2}{n^4}\cdot T_{n-r,2}(x_0) + \frac{(rx_0)^4}{n^4}\right] \ +\ x_0\left[\frac{1}{n^4}\cdot T_{n-r,4}(x_0) -\right.$$

$$\left.\left. -\ 4\cdot\frac{r(1-x_0)}{n^4}\cdot T_{n-r,3}(x_0) + 6\cdot\frac{(r(1-x_0))^2}{n^4}\cdot T_{n-r,2}(x_0) + \frac{(r(1-x_0))^4}{n^4}\right]\right\}$$

$$= \frac{1}{M^2}\cdot\left(\frac{n}{\varphi(x_0)^2}\right)^2\cdot\left\{\frac{1}{n^4}\cdot T_{n-r,4}(x_0) - 8\cdot\frac{r}{n^4}\cdot\varphi(x_0)^2\cdot T_{n-r,3}(x_0) +\right.$$

$$+ 6 \cdot \frac{r^2}{n^4} \cdot \varphi(x_0)^2 \cdot T_{n-r,2}(x_0) + \frac{r^4}{n^4} \cdot \varphi(x_0)^2 (1 - 3\varphi(x_0)^2) \Big\} .$$

Hence, by [1, p. 128, Lemma 9.4.4] and (28) we obtain
$\frac{n}{\varphi(x_0)^2} \cdot \left(L_{n,r} g_{M,n,x_0}\right)(x_0) \ \le$

$$\begin{aligned}
&\le \ \frac{1}{M^2} \cdot \left(\frac{n}{\varphi(x_0)^2}\right)^2 \cdot \frac{C}{n^4} \cdot \Big\{ (n-r)^2 \varphi(x_0)^4 + 8r\varphi(x_0)^2 \cdot (T_{n-r,6}(x_0))^{1/2} \ + \\
&+ \ 6r^2 \cdot \varphi(x_0)^2 (n-r) \varphi(x_0)^2 + r^4 \varphi(x_0)^2 (1 + 3\varphi(x_0)^2) \Big\} \\
&\le \ \frac{C}{M^2} \cdot \frac{1}{n^2 \varphi(x_0)^2} \cdot \Big\{ n^2 \varphi(x_0)^4 + 8 r^2 n^{3/2} \varphi(x_0)^5 \ + \\
&+ \ 6r^2 \cdot n\varphi(x_0)^4 + r^4 \varphi(x_0)^2 + 3r^4 \varphi(x_0)^4 \Big\} \ \le \ \frac{C}{M^2} \ \to \ 0
\end{aligned}$$

as $M \to \infty$. (Here $C > 0$ denotes an absolute constant which can depends only on r and it can be different at each occurrence).

Remark 3. *For $n > 2r$ we have*

$$\frac{1}{\sqrt{n}} \ \le \ \frac{\sqrt{n + r(r-1)}}{n} \ \le \ \sqrt{\frac{r+1}{2n}} \cdot \frac{1}{\sqrt{n}}$$

Therefore, by Corollary 2 we have for $\phi \equiv \varphi$ the following direct result:

$$\|L_{n,r} f - f\| \ \le \ C \ \omega_\phi^2 \left(f, \frac{1}{\sqrt{n}} \right) . \tag{29}$$

The constant C may depends only on φ, ϕ and r.

Thus the next theorem will constitute an inverse of type B for (29) :

Theorem 4. *Let $(L_{n,r})_{n>2r}$ be given by (1) and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Then there exist two constant K and $\tilde{C}$ ($\tilde{C}$ may depends only on φ, ϕ andr) such that for all $f \in C[0,1]$ and m, n with $M \ge Kn$ we have*

$$\omega_\varphi^2 \left(f, \frac{1}{\sqrt{n}} \right) \ \le \ \tilde{C} \cdot \frac{m}{n} \cdot \left(\|L_{n,r} f - f\| \ + \ \|L_{m,r} f - f\| \right) \tag{30}$$

Proof. Using (3), Lemma 3 : (4) - (5), Lemma 6 : (21) - (22) and Lemma 7, we obtain (30) in view of [9, p. 372, Theorem 1].

3. **A new generalized Brass operator**

In this section we establish direct and converse theorems for the operators defined by (2).

Theorem 5. *Let $(L^\alpha_{n,r})_{n>2r}$ be given by (2) and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. Then there exists an absolute constant $C > 0$ such that for all $f \in C[0,1]$ we have*

$$\|L_{n,r} f - f\| \ \le \ C \ \omega_\varphi^2 \left(\ f \ , \ \sqrt{\frac{1}{1+\alpha} \cdot \left(\frac{n + r(r-1)}{n^2} + \alpha \right)} \ \right)$$

Proof. By [5, p. 1180, Lemma 3.1] we have for $\alpha > 0$ and $x \in (0,1)$ the following identity

$$w_{n-r,k}(x,\alpha) \;=\; \binom{n-r}{k} \cdot \frac{B\left(x\alpha^{-1}+k,(1-x)\alpha^{-1}+n-r-k\right)}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}.$$

Consequently, $L_{n,r}^{\alpha} f$ can be represented by means of the operator (1), as follows

$$(L_{n,r}^{\alpha} f)(x) \;=\; \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot (L_{n,r} f)(t)\; dt \qquad (31)$$

On the other hand, by (31) and [8, p. 214, Theorem 2.1] we have

$$L_{n,r}^{\alpha}(u-x,x) \;=$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot L_{n,r}(u-x,t)\; dt$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \left[L_{n,r}^{\alpha}(u-t,t) + L_{n,r}(t-x,x)\right]\; dt$$

$$= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\; (t-x)\; dt \;=\; 0 \qquad (32)$$

and

$$\begin{aligned}
L_{n,r}^{\alpha}((u-x)^2,x) \;&= \\
&= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot L_{n,r}((u-x)^2,t)\; dt \\
&= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \big[L_{n,r}((u-t)^2,t) + \\
&+ 2(t-x)L_{n,r}(u-t,t) + (t-x)^2\big]\; dt \\
&= \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot \frac{n+r(r-1)}{n^2} \cdot t(1-t)\; dt \;+ \\
&+ \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} \cdot (t-x)^2\; dt \\
&= \frac{1}{1+\alpha} \cdot \left(\frac{n+r(r-1)}{n^2} + \alpha\right) \cdot \varphi(x)^2 \qquad (33)
\end{aligned}$$

Furthermore, by (3)

$$|(L_{n,r}^{\alpha} f)(x)| \;\le\; \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)} \cdot |(L_{n,r} f)(t)|\; dt \;\le\; \|f\|.$$

So

$$\|L_{n,r}^{\alpha} f\| \;\le\; \|f\| \qquad (34)$$

for all $f \in C[0,1]$. Now, using (32), (33), (34) and the standard method [1, Chap. 9], we obtain the assertion of the theorem.

In what follows we shall use some lemmas. These are the following:

Lemma 4. *For* $(L_{n,r})_{n>2r}$, $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$ *and* $f\in C[0,1]$ *we have*

$$\frac{1}{n}\cdot\|\varphi^2(L_{n,r}f)''\| \;\le\; C_0\;\left(\|L_{n,r}f-f\| \;+\; \|L_{Kn,r}f-f\|\right),$$

where $C_0>0$ *is an absolute constant.*

Proof. The announced inequality is the estimate (14) for $m=Kn$ given in [9, p. 373], using the estimates (4), (5), (21), (22) and Lemma 7.

Lemma 5. *For* $(L_{n,r})_{n>2r}$, $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$ *and* $f\in C[0,1]$ *we have*

$$\|L^{\alpha}_{n,r}f-L_{n,r}f\| \;\le\; \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r}f)''\|.$$

Proof. By (31) and Taylor's formula :

$$(L_{n,r}f)(t)=(L_{n,r}f)(x)+(t-x)(L_{n,r}f)'(x)+\int_x^t\,(t-u)(L_{n,r}f)''(u)\;du$$

we have

$$\begin{aligned}
(L^{\alpha}_{n,r}f)(x)-(L_{n,r}f)(x) \;&=\; \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1\,t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\
&\cdot\;\left[(t-x)(L_{n,r}f)'(x)+\int_x^t\,(t-u)(L_{n,r}f)''(u)\;du\right]\;dt\\
&=\; \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1\,t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\
&\cdot\;\left\{\int_x^t\,(t-u)(L_{n,r}f)''(u)\;du\right\}\;dt. \qquad (35)
\end{aligned}$$

Hence, by [1, p. 140, Lemma 9.6.1] we obtain

$$\begin{aligned}
|(L^{\alpha}_{n,r}f)(x)-(L_{n,r}f)(x)| \;&=\; \frac{1}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1\,t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\
&\cdot\;\Big|\int_x^t\frac{|t-u|}{u(1-u)}\cdot u(1-u)\;|(L_{n,r}f)''(u)|\;du\;\Big|\;\;dt\\
&\le\; \frac{\|\varphi^2(L_{n,r}f)''\|}{B\left(x\alpha^{-1},(1-x)\alpha^{-1}\right)}\cdot\int_0^1\,t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}\cdot\\
&\cdot\;\frac{(t-x)^2}{x(1-x)}\;dt\;=\;\frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r}f)''\;Vert,
\end{aligned}$$

which was to be proved.

We have the following result:

Theorem 6. *Let* $(L^{\alpha}_{n,r})_{n>2r}$ *be given by (2) and* $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$. *If* $\alpha=\alpha(n)$ *and* $(\alpha/(1+\alpha))\cdot n\,(C_0+C_0\cdot C_1(r)+4K)\;\le\;\tilde{\alpha}\;<\;1$ *then*

$$\begin{aligned}
(1-\tilde{\alpha})\left(\|L_{n,r}f-f\|+\|L_{Kn,r}f-f\|\right) \;&\le\; \|L^{\alpha}_{n,r}f-f\|+\|L^{\alpha}_{Kn,r}f-f\| \;\le\\
&\le\; (1+\tilde{\alpha})\left(\|L_{n,r}f-f\|+\|L_{Kn,r}f-f\|\right)
\end{aligned}$$

for all $f \in C[0,1]$. Moreover, there exists an absolute constant $C > 0$ such that for all $f \in C[0,1]$ we have

$$C^{-1}\ \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right) \ \le\ \|L_{n,r}^\alpha f - f\| + \|L_{Kn,r} f - f\| \ \le\ C\ \omega_\varphi^2\left(f, \frac{1}{\sqrt{n}}\right).$$

Proof. We have, in view of Lemma 11 :

$$\begin{aligned}
&\|L_{n,r}^\alpha f - f\| + \|L_{Kn,r}^\alpha f - f\| \ \le \\
&\le\ \|L_{n,r}^\alpha f - L_{n,r} f\| + \|L_{n,r} f - f\| + \|L_{Kn,r}^\alpha f - L_{Kn,r} f - f\| + \|L_{Kn,r} f - f\| \\
&\le\ \frac{\alpha}{1+\alpha}\cdot\|\varphi^2 (L_{n,r} f)''\| + \|L_{n,r} f - f\| + \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{Kn,r} f)''\| + \|L_{Kn,r} f - f\|.
\end{aligned}$$

Using Lemma 3 : (4) - (5), we obtain

$$\begin{aligned}
\|\varphi^2(L_{Kn,r} f)''\| &\le \|\varphi^2(L_{Kn,r}(f - L_{n,r} f))''\| + \|\varphi^2(L_{Kn,r}(L_{n,r} f))''\| \\
&\le 4Kn\|f - L_{n,r} f\| + C_1(r)\cdot\|\varphi^2(L_{n,r} f)''\|.
\end{aligned}$$

Thus

$$\|L_{n,r}^\alpha f - f\| + \|L_{Kn,r}^\alpha f - f\| \le \frac{\alpha}{1+\alpha}\cdot(1 + C_1(r))\cdot\|\varphi^2(L_{n,r} f)''\| +$$

$$+\left(\frac{\alpha}{1+\alpha}\cdot 4Kn + 1\right)\cdot\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|.$$

Hence, by Lemma 10 we obtain

$$\begin{aligned}
&\|L_{n,r}^\alpha f - f\| + \|L_{Kn,r} f - f\| \ \le \\
&\le\ \frac{\alpha}{1+\alpha}\cdot nC_0\cdot(1 + C_1(r))\cdot(\ \|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|\ )\ + \\
&+\ \left(\frac{\alpha}{1+\alpha}\cdot 4Kn + 1\right)\cdot\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\| \\
&=\ \left[1 + \frac{\alpha}{1+\alpha}\cdot(nC_0(1 + C_1(r)) + 4K)\right]\cdot\|L_{n,r} f - f\|\ + \\
&+\ \left[1 + \frac{\alpha}{1+\alpha}\cdot nC_0(1 + C_1(r))\right]\cdot\|L_{Kn,r} f - f\| \\
&\le\ (1 + \tilde{\alpha})\cdot(\ \|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|\ ) \qquad (36)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\|L_{n,r} f - f\| + \|L_{Kn,r} f - f\| \ \le \\
&\le\ \|L_{n,r}^\alpha f - L_{n,r} f\| + \|L_{n,r}^\alpha f - f\| + \|L_{Kn,r}^\alpha f - L_{Kn,r} f\| + \|L_{Kn,r}^\alpha f - f\| \\
&\le\ \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{n,r} f)''\| + \|L_{n,r}^\alpha f - f\| + \frac{\alpha}{1+\alpha}\cdot\|\varphi^2(L_{Kn,r} f)''\| + \|L_{Kn,r}^\alpha f - f\|.
\end{aligned}$$

Using Lemma 10 and Lemma 3 : (4) - (5), we obtain

$$\|L_{n,r} f - f\| + \|L_{kn,r} f - f\| \ \le$$

$$\le \frac{\alpha}{1+\alpha}\cdot nC_0\cdot(\ \|L_{n,r} f - f\| + \|L_{Kn,r} f - f\|\ ) + \|L_{n,r}^\alpha f - f\| +$$

$$+\frac{\alpha}{1+\alpha}\cdot\big(\ 4Kn\|L_{n,r} f - f\| + C_1(r)\|\varphi^2(L_{n,r} f)''\|\ \big) + \|L_{Kn,r}^\alpha f - f\|$$

$$\leq \|L^{\alpha}_{n,r}f - f\| + \|L^{\alpha}_{Kn,r}f - f\| + \frac{\alpha}{1+\alpha} \cdot (nC_0(1 + C_1(r)) + 4Kn) \cdot \|L_{n,r}f - f\| +$$

$$+\frac{\alpha}{1+\alpha} \cdot nC_0(1 + C_1(r)) \cdot \|L_{Kn,r}f - f\|$$

$$\leq \|L^{\alpha}_{n,r}f - f\| + \|L^{\alpha}_{Kn,r}f - f\| + \tilde{\alpha} \left(\|L_{n,r}f - f\| + \|L_{Kn,r}f - f\| \right).$$

Hence

$$(1 - \tilde{\alpha}) \left(\|L_{n,r}f - f\| + \|L_{Kn,r}f - f\| \right) \leq \|L^{\alpha}_{n,r}f - f\| + \|L^{\alpha}_{Kn,r}f - f\|. \quad (37)$$

In conclusion (36) and (37) imply the assertion of the theorem. Moreover, by (29) and (30), we obtain the second statement of the theorem using the first one.

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