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ASYMPTOTIC FIXED POINT THEOREMS IN E-METRIC SPACES

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Abstract. In this paper we prove two asymptotical fixed point theorems in E-metric spaces. The first theorem is a variant of Ciric-Reich-Rus theorem in E-metric space, the next theorem is the asymptotic variant of this theorem.

Let E be a real linear space partially ordered by $\leq$ and let $E_+ = \{e \in E : e \geq 0\}$ be the positive cone of E. We consider on E a linear convergence, i.e., a convergence with: ([1])

1.) if $e_n = e$, $\forall n \in \mathbb{N} \Rightarrow \lim e_n = e$.
2.) if $\lim e_n = e$ implies $\lim e_{n'} = e$ for every subsequence $(e_{n'})$ of (e_n).
3.) $\lim e_n = e$ and $\lim f_n = f$ imply $\lim(e_n + f_n) = e + f$.
4.) $\lim e_n = e$ implies $\lim(r \cdot e_n) = r \cdot e$, $\forall r \in \mathbb{R}$.
5.) if $e_n \leq f_n$, $\forall n \in \mathbb{N}$ and $\lim e_n = e$, $\lim f_n = f$ then $e \leq f$.
6.) if $e_n \leq f_n \leq g_n$, $\forall n \in \mathbb{N}$ and $\lim e_n = \lim g_n = e$ then also $\lim f_n = e$.

Let X be a nonempty set and let E be an ordered linear space with a linear convergence. An *E-metric on* X is a mapping $d : X \times X \longrightarrow E_+$ subject to the usual axioms:

1.) $d(x, y) = 0_E$ if and only if $x = y$.
2.) $d(x, y) = d(y, x)$, $\forall x, y \in X$.
3.) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

By *E-metric space* we mean a nonempty set X with an E-metric on X. The ordered space E is briefly called the *metrizing space* for X.

A sequence (x_n) of elements of an E-metric space X is said *convergent toward* $x \in X$ (and we write $x_n \to x$) if $d(x_n, x) \to 0$ as $n \to \infty$.

A sequence (x_n) in X is said to be a *Cauchy sequence* if $d(x_n, x_m) \to 0$, $n, m \to \infty$.

The E-metric space X is said to be *sequentially complete*, if each Cauchy sequence in X converges to a point in X.

A subset Y of an E-metric space X is said to be bounded if the set $\{d(x, y) : x, y \in Y\}$ has an upper bound in E.

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We note $F_A := \{x \in X | A(x) = x\}$ - the fixed point set of A.

In this note we need the following results:

Theorem 1. (Ciric - Reich -Rus, [2], [3]**)** *Let (X, d) be a complet metric space and $A : X \longrightarrow X$ be an operator. Suppose that there exists the numbers a, b, c such that $0 \leq a + b + c < 1$, and let A be a map such that:*

$$d(A(x), A(y)) \leq a \cdot d(x, A(x)) + b \cdot d(y, A(y)) + c \cdot d(x, y), \quad \forall x,\, y \in X.$$

Then A has an unique fixed point.

Lemma 1. ([3]**)** *Let X be a nonempty set and $f : X \longrightarrow X$ a mapping. If there exists $k \in \mathbb{N}$ such that $F_{f^k} = \{x^*\}$, then $F_f = \{x^*\}$.*

The first result is the generalization of Theorem 1 in E-metric spaces:

Theorem 2. *Let X be a sequentially complete E-metric space. Let $S, T, R : E_+ \longrightarrow E_+$ are increasing operators and let $A : X \longrightarrow X$ be an operator which satisfy the following condition:*

$$d(A(x), A(y)) \leq S\, d(x, A(x)) + T\, d(y, A(y)) + R\, d(x, y), \quad x,\, y \in X.$$

Suppose that

(i) *$1_E - T$ is a bijection*

(ii) *there exists $x_0 \in X$ such that, $\sum_{n\in\mathbb{N}} [(1_E - T)^{-1}(S + R)]^n d(x_0, A(x_0))$ converges.*

Then A has an unique fixed point.

Proof. Let $y = A(x)$. Then

$$d(A(x), A^2(x)) \leq S\, d(x, A(x)) + T\, d(A(x), A^2(x)) + R\, d(x, A(x))$$

and we have that

$$(1_E - T)\, d(A(x), A^2(x)) \leq (S + R)\, d(x, A(x)).$$

Because $(1_E - T)$ is a bijection, we have

$$d(A(x), A^2(x)) \leq (1_E - T)^{-1} \cdot (S + R)\, d(x, A(x))$$

$$\dots$$

$$d(A^{n+1}(x), A^n(x)) \leq (1_E - T)^{-1}(S + R)\, d(A^n(x), A^{n-1}(x)) \leq \dots$$
$$\dots \leq [(1_E - T)^{-1}(S + R)]^n\, d(x, A(x)), \quad \forall n \in \mathbb{N}^*.$$

We want to prove that the $(A^n x_0)_n$ is a Cauchy sequence:

$$d(A^{n+m}(x_0), A^n(x_0)) \leq d(A^{n+m}(x_0), A^{n+m-1}(x_0)) +$$
$$+ d(A^{n+m-1}(x_0), A^{n+m-2}(x_0)) + ... + d(A^{n+1}(x_0), A^n(x_0)) \leq$$
$$\leq [(1_E - T)^{-1}(S + R)]^{n+m-1}\, d(x_0, A(x_0)) +$$
$$+ [(1_E - T)^{-1}(S + R)]^{n+m-2}\, d(x_0, A(x_0)) + \dots$$
$$\dots + [(1_E - T)^{-1}(S + R)]^n\, d(x_0, A(x_0)) \to 0. \qquad \text{(ii)}$$

Because the sequence is Cauchy and X is sequencially complete we have that the sequence is convergent and let $x^* = \lim A^n x_0$.

We have

$$d(x^*, A(x^*)) \le d(x^*, A^n(x_0)) + d(A^n(x_0), A(x^*)) \le$$
$$\le d(x^*, A^n(x_0)) + S\,d(A^{n-1}(x_0), A^n(x_0)) + T\,d(x^*, A(x^*)) + R\,d(A^{n-1}(x_0), x^*).$$

Hence

$$(1_E - T)d(x^*, A(x^*)) \le d(x^*, A^n(x_0)) + S\,d(A^{n-1}(x_0), A^n(x_0)) +$$
$$+R\,d(A^{n-1}(x_0), x^*)).$$
$$d(x^*, A(x^*)) \le (1_E - T)^{-1}[d(x^*, A^n(x_0)) + S\,d(A^{n-1}(x_0), A^n(x_0)) +$$
$$+R\,d(A^{n-1}(x_0), x^*))] \to 0.$$

By letting $n \to \infty$ we have $d(x^*, A(x^*)) = 0$, i.e. $F_A = \{x^*\}$ and $A^n(x_0) \to x^*$. □

The following theorem is the asymptotic variant of the Theorem 1 in E-metric spaces.

Theorem 3. *Let X be a sequentially complete E-metric space. Let $S, T, R : E_+ \longrightarrow E_+$ and let $A : X \longrightarrow X$ be a map for which there exists $k \in \mathbb{N}^*$ such that*

$$d(A^k x, A^k y) \le S\,d(x, A^k x) + T\,d(y, A^k y) + R\,d(x, y), \quad \forall x,\, y \in X.$$

Suppose that:

(i) *$1_E - T$ is a bijection*

(ii) *there exists $x_0 \in X$ such that, $\sum_{n \in N} [(1_E - T)^{-1}(S + R)]^n d(x_0, A^k(x_0))$ converges.*

Then A has an unique fixed point.

Proof. We apply the Theorem 2 for the iterate A^k and we have that A^k has an unique fixed point. Now apply the lemma and we have that the operator A has an unique fixed point.

Remarks.

1. When $S = 0$ and $T = 0$, we have the asymptotic variant of Banach fixed point theorem in E-metric spaces [1].
2. Let $E = \mathbb{R}^n$ then we have an asymptotic variant of Perov fixed point theorem ([3]).

References

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