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ON CERTAIN CLASSES OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS. II

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Abstract. The object of the present paper is to obtain modified Hadamard products (convolutions) of several functions belonging to the classes $T^*(p,\alpha)$ and $C(p,\alpha)$ consisting of analytic and p-valent functions with negative coefficients. We also obtain class preserving integral operator of the form

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt, \quad c > -p$$

for the classes $T^*(p,\alpha)$ and $C(p,\alpha)$. Conversely, when F belongs to $T^*(p,\alpha)$ and $C(p,\alpha)$, radii of p-valence of f defined by the above equation are obtained.

1. Introduction

Let $S(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (p \in \mathbb{N} = \{1, 2, \cdots\})$$

which are analytic and p-valent in the unit disc $\mathrm{U} = \{z : |z| < 1\}$. A function f of $S(p)$ is called p-valent starlike of order α if f satisfies the following conditions

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathrm{U} \tag{1.1}$$

and

$$\int_0^{2\pi} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi$$

for $0 \le \alpha < p$, $p \in \mathbb{N}$ and $z \in \mathrm{U}$. We denote by $S^*(p,\alpha)$ the class of all p-valent starlike functions of order α. The class $S^*(p,\alpha)$ was studied by Patil and Thakare [3]. Further a function f of $S(p)$ is called p-valent convex of order α if f satisfies the following conditions

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathrm{U} \tag{1.2}$$

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and

$$\int_0^{2\pi} \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi$$

for $0 \le \alpha < p$, $p \in \mathbb{N}$ and $z \in \mathrm{U}$. We denote by $K(p,\alpha)$ the class of all p-valent convex functions of order α.

It follows from (1.1) and (1.2) that

$$f(z) \in K(p,\alpha) \text{ if and only if } zf'(z)/p \in S^*(p,\alpha), \quad 0 \le \alpha < p.$$

Let $T(p)$ denote the subclass of $S(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_{p+n} \ge 0;\ p,\, n \in \mathbb{N}). \tag{1.3}$$

We denote by $T^*(p,\alpha)$ and $C(p,\alpha)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ with $T(p)$, that is $T^*(p,\alpha) = S^*(p,\alpha) \cap T(p)$ and $C(p,\alpha) = K(p,\alpha) \cap T(p)$.

The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were studied by Owa [2].

In order to prove our results for functions belonging to the classes $T^*(p,\alpha)$ and $C(p,\alpha)$ we shall require the following lemmas given by Owa [2] and Aouf [1].

Lemma 1.1. *Let the function f be defined by (1.3); then $f \in T^*(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (p+n-\alpha) a_{p+n} \le p - \alpha.$$

The result is sharp for the functions

$$f(z) = z^p - \frac{p-\alpha}{p+n-\alpha} z^{p+n}, \quad n \in \mathbb{N}. \tag{1.4}$$

Lemma 1.2. *Let the function f be defined by (1.3); then $f \in C(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (p+n)(p+n-\alpha)\, a_{p+n} \le p(p-\alpha).$$

The result is sharp for the functions

$$f(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} z^{p+n}, \quad n \in \mathbb{N}.$$

2. **Modified Hadamard product**

Let the functions f_i be defined, for $i \in \{1,\, 2,\, 3\}$, by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,i}\, z^{p+n} \quad (a_{p+n,i} \ge 0). \tag{2.1}$$

The modified Hadamard product (convolution) of f_1 and f_2 is defined here by

$$f_1 * f_2(z) = z^p - \sum_{n=1}^{\infty} a_{p+n,1}\, a_{p+n,2}\, z^{p+n}.$$

Theorem 2.1. *Let the functions* f_i*,* $i \in \{1, 2\}$*, defined by (2.1) be in the class* $T^*(p,\alpha)$*. Then* $f_1 * f_2(z) \in T^*(p, \beta(p,\alpha))$*, where*

$$\beta(p,\alpha) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2 - (p-\alpha)^2}. \tag{2.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(p,\alpha)$ such that

$$\sum_{n=1}^{\infty} \frac{p+n-\beta}{p-\beta}\, a_{p+n,1}\, a_{p+n,2} \leq 1.$$

Since

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,1} \leq 1 \ \text{ and } \ \sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,2} \leq 1,$$

by the Cauchy-Schwarz inequality we have

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha} \sqrt{a_{p+n,1}\, a_{p+n,2}} \leq 1.$$

Thus it is sufficient to show that

$$\frac{p+n-\beta}{p-\beta}\, a_{p+n,1}\, a_{p+n,2} \leq \frac{p+n-\alpha}{p-\alpha} \sqrt{a_{p+n,1}\, a_{p+n,2}} \quad (n \geq 1),$$

that is

$$\sqrt{a_{p+n,1}\, a_{p+n,2}} \leq \frac{(p-\beta)(p+n-\alpha)}{(p-\alpha)(p+n-\beta)}.$$

Note that

$$\sqrt{a_{p+n,1}\, a_{p+n,2}} \leq \frac{p-\alpha}{p+n-\alpha} \quad (n \geq 1).$$

Consequently, we need only to prove that

$$\frac{p-\alpha}{p+n-\alpha} \leq \frac{(p-\beta)(p+n-\alpha)}{(p-\alpha)(p+n-\beta)} \quad (n \geq 1)$$

or, equivalently, that

$$\beta \leq p - \frac{n\,(p-\alpha)^2}{(p+n-\alpha)^2 - (p-\alpha)^2}, \quad (n \geq 1).$$

Since

$$\Psi(n) = p - \frac{n\,(p-\alpha)^2}{(p+n-\alpha)^2 - (p-\alpha)^2}, \quad (n \geq 1), \tag{2.3}$$

is an increasing function of n $(n \geq 1)$, letting $n = 1$ in (2.3) we obtain

$$\beta \leq \Psi(1) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2 - (p-\alpha)^2},$$

which completes the proof of Theorem 1.

Finally, by taking the functions

$$f_i(z) = z^p - \frac{p-\alpha}{p+1-\alpha}\, z^{p+1}, \quad (i \in \{1,\, 2\}), \tag{2.4}$$

we can see that the result is sharp.

In a similar manner, with the aid of Lemma 1.2, we can prove

Theorem 2.2. *Let the functions f_i, $i \in \{1,\, 2\}$, defined by (2.1) be in the class $C(p,\alpha)$. Then $f_1 * f_2(z) \in C(p, \beta(p,\alpha))$, where*

$$\beta(p,\alpha) = p - \frac{(p-\alpha)^2}{(p+1-\alpha)^2(p+1)/p - (p-\alpha)^2}.$$

The result is sharp for the functions

$$f_i(z) = z^p - \frac{p\,(p-\alpha)}{(p+1)(p+1-\alpha)}\, z^{p+1}, \;\; i \in \{1,\, 2\}. \tag{2.5}$$

Theorem 2.3. *Let the function f_1 defined by (2.1) be in the class $T^*(p,\alpha)$ and let the function f_2 defined by (2.1) be in the class $T^*(p,\gamma)$; then $f_1 * f_2(z) \in T^*(p, \zeta(p,\alpha,\gamma))$, where*

$$\zeta(p,\alpha,\gamma) = p - \frac{(p-\alpha)(p-\gamma)}{(p+1-\alpha)(p+1-\gamma) - (p-\alpha)(p-\gamma)}.$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 2.1, we get

$$\zeta \le \Phi(n) = p - \frac{n\,(p-\alpha)(p-\gamma)}{(p+n-\alpha)(p+n-\gamma) - (p-\alpha)(p-\gamma)}. \tag{2.6}$$

Since the function $\Phi(n)$ is an increasing function of n $(n \ge 1)$, letting $n = 1$ in (2.6) we obtain

$$\zeta \le \Phi(1) = p - \frac{(p-\alpha)(p-\gamma)}{(p+1-\alpha)(p+1-\gamma) - (p-\alpha)(p-\gamma)},$$

which evidently proves Theorem 2.3.

Further, taking

$$f_1(z) = z^p - \frac{p-\alpha}{p+1-\alpha}\, z^{p+1} \;\text{ and }\; f_2(z) = z^p - \frac{p-\gamma}{p+1-\gamma}\, z^{p+1}. \tag{2.7}$$

Theorem 2.4. *Let the function f_1 defined by (2.1) be in the class $C(p,\alpha)$ and the function f_2 defined by (2.1) be in the class $C(p,\gamma)$; then $f_1 * f_2(z) \in C(p, \zeta(p,\alpha,\gamma))$, where*

$$\zeta(p,\alpha,\gamma) = p - \frac{(p-\alpha)(p-\gamma)}{(p+1-\alpha)(p+1-\gamma)(p+1)/p - (p-\alpha)(p-\gamma)}.$$

The result is sharp for the functions

$$f_1(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}\, z^{p+1} \;\text{ and }\; f_2(z) = z^p - \frac{p(p-\gamma)}{(p+1-\gamma)(p+1)}\, z^{p+1}.$$

Corollary 2.1. *Let the functions f_i, $i \in \{1, 2, 3\}$), defined by (2.1) be in the class $T^*(p,\alpha)$; then $f_1 * f_2 * f_3(z) \in T^*(p,\eta(p,\alpha))$, where*

$$\eta(p,\alpha) = p - \frac{(p-\alpha)^3}{(p+1-\alpha)^3 - (p-\alpha)^3}.$$

The result is the best possible for the functions

$$f_i(z) = z^p - \frac{p-\alpha}{p+1-\alpha}\, z^{p+1}, \quad i \in \{1, 2, 3\}).$$

Proof. From Theorem 2.1 we have $f_1 * f_2(z) \in T^*(p,\beta(p,\alpha))$, where β is given by (2.2). We use now Theorem 2.3 and we get $f_1 * f_2 * f_3(z) \in T^*(p,\eta(p,\alpha))$, where

$$\eta(p,\alpha) = p - \frac{(p-\alpha)(p-\beta)}{(p+1-\alpha)(p+1-\beta) - (p-\alpha)(p-\beta)} = p - \frac{(p-\alpha)^3}{(p+1-\alpha)^3 - (p-\alpha)^3}.$$

This completes the proof of Corollary 2.1.

Corollary 2.2. *Let the functions f_i, $i \in \{1, 2, 3\}$), defined by (2.1) be in the class $C(p,\alpha)$; then $f_1 * f_2 * f_3(z) \in C(p,\eta(p,\alpha))$, where*

$$\eta(p,\alpha) = p - \frac{(p-\alpha)^3}{(p+1-\alpha)^3(p+1)^2/p^2 - (p-\alpha)^3}.$$

The result is the best possible for the functions

$$f_i(z) = z^p - \frac{p(p-\alpha)}{(p+1-\alpha)(p+1)}\, z^{p+1}, \quad i \in \{1, 2, 3\}).$$

Theorem 2.5. *Let the function f_1 defined by (2.1) be in the class $T^*(p,\alpha)$ and the function f_2 defined by (2.1) be in the class $T^*(p,\gamma)$; then $f_1 * f_2(z) \in C(p,\beta(p,\alpha,\gamma))$, where*

$$\beta(p,\alpha,\gamma) = p - \frac{(p+1)(p-\alpha)(p-\gamma)}{p(p+1-\alpha)(p+1-\gamma) - (p+1)(p-\alpha)(p-\gamma)}.$$

The result is sharp.

Proof. Since $f_1 \in T^*(p,\alpha)$ and $f_2 \in T^*(p,\gamma)$, we have

$$\sum_{n=1}^{\infty}(p+n-\alpha)\, a_{p+n,1} \le p-\alpha \quad \text{and} \quad \sum_{n=1}^{\infty}(p+n-\gamma)\, a_{p+n,2} \le p-\gamma.$$

It follows that

$$\sum_{n=1}^{\infty}(p+n-\alpha)(p+n-\gamma)\, a_{p+n,1}a_{p+n,2} \le (p-\alpha)(p-\gamma).$$

We want to find the largest $\beta = \beta(p,\alpha,\gamma)$ such that

$$\sum_{n=1}^{\infty}(p+n-\beta)(p+n)\, a_{p+n,1}a_{p+n,2} \le p(p-\beta).$$

This will be certainly satisfied if

$$\frac{(p+n-\beta)(p+n)}{p(p-\beta)} \leq \frac{(p+n-\alpha)(p+n-\gamma)}{(p-\alpha)(p-\gamma)} \quad (n \geq 1),$$

or

$$\beta \leq p - \frac{n(p+n)(p-\alpha)(p-\gamma)}{p(p+n-\alpha)(p+n-\gamma)-(p+n)(p-\alpha)(p-\gamma)} \quad (n \geq 1).$$

Since

$$K(n) = p - \frac{n(p+n)(p-\alpha)(p-\gamma)}{p(p+n-\alpha)(p+n-\gamma)-(p+n)(p-\alpha)(p-\gamma)} \quad (n \geq 1) \tag{2.8}$$

is an increasing function of n $(n \geq 1)$, letting $n = 1$ in (2.8) we obtain

$$\beta \leq K(1) = p - \frac{(p+1)(p-\alpha)(p-\gamma)}{p(p+1-\alpha)(p+1-\gamma)-(p+1)(p-\alpha)(p-\gamma)},$$

and so $\beta(p,\alpha,\gamma) = K(1)$. Finally, the result is sharp for the functions f_1 and f_2 defined by (2.7).

Theorem 2.6. *Let the functions* f_i, $i \in \{1, 2\}$, *defined by (2.1) be in the class* $T^*(p,\alpha)$; *then the function*

$$h(z) = z^p - \sum_{n=1}^{\infty} \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) z^{p+n} \tag{2.9}$$

belongs to the class $T^*(p,\delta(p,\alpha))$, *where*

$$\delta(p,\alpha) = p - \frac{2(p-\alpha)^2}{(p+1-\alpha)^2 - 2(p-\alpha)^2}.$$

The result is sharp.

Proof. By virtue of Lemma 1.1, we obtain

$$\sum_{n=1}^{\infty} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 a_{p+n,1}^2 \leq \left\{\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,1}\right\}^2 \leq 1 \tag{2.10}$$

and

$$\sum_{n=1}^{\infty} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 a_{p+n,2}^2 \leq \left\{\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n,2}\right\}^2 \leq 1. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2 \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) \leq 1.$$

Therefore, we need to find the largest δ such that

$$\frac{p+n-\delta}{p-\delta} \leq \frac{1}{2} \left\{\frac{p+n-\alpha}{p-\alpha}\right\}^2,$$

that is

$$\delta \leq p - \frac{2n(p-\alpha)^2}{(p+n-\alpha)^2 - 2(p-\alpha)^2} \quad (n \geq 1).$$

Since

$$D(n) = p - \frac{2n(p-\alpha)^2}{(p+n-\alpha)^2 - 2(p-\alpha)^2} \quad (n \geq 1)$$

is an increasing function of n $(n \geq 1)$, we readily have

$$\delta \leq D(1) = p - \frac{2(p-\alpha)^2}{(p+1-\alpha)^2 - 2(p-\alpha)^2}.$$

The result is sharp for the functions f_i, $i \in \{1, 2\}$ given by (2.4).

Theorem 2.7. *Let the functions* f_i, $i \in \{1, 2\}$, *defined by (2.1) be in the class* $C(p, \alpha)$*; then the function* $h(z)$ *defined by (2.9) belongs to the class* $C(p, \delta(p, \alpha))$*, where*

$$\delta(p, \alpha) = p - \frac{2p(p-\alpha)^2}{(p+1)(p+1-\alpha)^2 - 2p(p-\alpha)^2}.$$

The result is sharp for the functions f_i, $i \in \{1, 2\}$ *defined by (2.5).*

3. **Integral operators**

Theorem 3.1. *Let the function* f *defined by (1.3) be in the class* $T^*(p, \alpha)$ *and let* d *be a real number such that* $d > -p$*; then the function* F *defined by*

$$F(z) = \frac{d+p}{z^d} \int_0^z t^{d-1} f(t)\, dt \tag{3.1}$$

also belongs to the class $T^*(p, \alpha)$.

Proof. From the representation of F it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad \text{where} \quad b_{p+n} = \frac{d+p}{d+p+n} a_{p+n}.$$

Therefore

$$\sum_{n=1}^{\infty} (p+n-\alpha) b_{p+n} = \sum_{n=1}^{\infty} (p+n-\alpha) \frac{d+p}{d+p+n} a_{p+n} \leq \sum_{n=1}^{\infty} (p+n-\alpha) a_{p+n} \leq p - \alpha,$$

since $f \in T^*(p, \alpha)$. Hence, by Lemma 1.1, $F \in T^*(p, \alpha)$.

Putting $d = 1 - p$ in Theorem 3.1 we get the following corollary.

Corollary 3.1. *Let the function* f *defined by (1.3) be in the class* $T^*(p, \alpha)$ *and let* F *be defined by*

$$F(z) = \frac{1}{z^{1-p}} \int_0^z \frac{f(t)}{t^p}\, dt;$$

then $F \in T^*(p, \alpha)$.

Theorem 3.2. *Let* d *be a real number such that* $d > -p$. *If* $F \in T^*(p, \alpha)$, *then the function* f *defined by (3.1) is p-valent in* $|z| < R_p^*$, *where*

$$R_p^* = \inf_n \left[\frac{p(p+n-\alpha)(d+p)}{(p+n)(p-\alpha)(d+p+n)} \right]^{1/n}.$$

The result is sharp.

Proof. Let $F(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}$ $(a_{p+n} \geq 0)$. It follows from (3.1) that

$$f(z) = \frac{z^{1-d}\left(z^d F(z)\right)'}{d+p} = z^p - \sum_{n=1}^{\infty} \frac{d+p+n}{d+p}\, a_{p+n} z^{p+n}.$$

To prove the result it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \quad \text{for} \quad |z| < R_p^*.$$

Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| = \left| - \sum_{n=1}^{\infty} (p+n) \frac{d+p+n}{d+p}\, a_{p+n} z^n \right| \leq \sum_{n=1}^{\infty} (p+n) \frac{d+p+n}{d+p}\, a_{p+n} \, |z|^n.$$

Thus $|f'(z)/z^{p-1} - p| \leq p$ if

$$\sum_{n=1}^{\infty} \frac{p+n}{p} \frac{d+p+n}{d+p}\, a_{p+n} \, |z|^n \leq 1. \tag{3.2}$$

But Lemma 1.1 confirm that

$$\sum_{n=1}^{\infty} \frac{p+n-\alpha}{p-\alpha}\, a_{p+n} \leq 1.$$

Thus (3.2) will be satisfied if

$$\frac{(p+n)(d+p+n)}{p(d+p)} |z|^n \leq \frac{p+n-\alpha}{p-\alpha} \quad (n \geq 1)$$

or if

$$|z| \leq \left[\frac{p(p+n-\alpha)(d+p)}{(p+n)(p-\alpha)(d+p+n)} \right]^{1/n} \quad (n \geq 1). \tag{3.3}$$

The required result follows now from (3.3). The result is sharp because the functions

$$f(z) = z^p - \frac{(p-\alpha)(d+p+n)}{(p+n-\alpha)(d+p)}\, z^{p+n} \quad (n \geq 1)$$

are defined by (3.1) when F are given by (1.4).

In a similar manner, with the aid of Lemma 1.2, we can prove the following theorem.

Theorem 3.3. *Let the function f defined by (1.3) be in the class $C(p, \alpha)$ and let d be a real number such that $d > -p$. Then the function F defined by (3.1) also belongs to the class $C(p, \alpha)$.*

Theorem 3.4. *Let d be a real number such that $d > -p$. If $F \in C(p, \alpha)$, then the function f defined by (3.1) is p-valent in $|z| < R_p^{**}$, where*

$$R_p^{**} = \inf_n \left[\frac{(p+n-\alpha)(d+p)}{(p-\alpha)(d+p+n)} \right]^{1/n}.$$

The result is sharp for the functions

$$f(z) = z^p - \frac{p(p-\alpha)(d+p+n)}{(p+n)(p+n-\alpha)(d+p)}\, z^{p+n} \quad (n \geq 1).$$

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