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ON A CLASS OF PARAMETRIC PARTIAL LINEAR COMPLEX VECTOR FUNCTIONAL EQUATIONS

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Abstract. In this paper one class of parametric complex vector partial linear functional equations is solved.

0. **Introduction**

First we introduce the following notations. Let $\mathcal{V}$, $\mathcal{V}'$ be finite dimensional complex vector spaces and $\mathbf{Z}_i$, $i \in \mathbf{N}$, be vectors in $\mathcal{V}$. We may assume that $\mathbf{Z}_i = (z_{i1}(t), \dots, z_{in}(t))^T$, where $z_{ij}(t)$ $(1 \le j \le n)$ are complex functions and $\mathbf{O} = (0, \dots, 0)^T$ is the zero-vector in $\mathcal{V}$ or $\mathcal{V}'$. We also denote by $\mathcal{V}^0$ the subspace of all real vectors in $\mathcal{V}$ (thus $\mathcal{V} = \mathcal{V}^0 + i\mathcal{V}^0$), and by $\mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ the space of linear mappings $\mathcal{V}^0 \to \mathcal{V}'$. Let (m, n) be the greatest common divisor of m and n.

In the present paper our object of investigation will be the following functional equation

$$\sum_{i=1}^{m+n} f_i\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j}, \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}\Big) = \mathbf{O} \tag{1}$$

$$(\mathbf{Z}_{m+n+i} \equiv \mathbf{Z}_i, \quad a \in \mathbf{C}),$$

where $\mathbf{C}$ is the field of complex numbers and $f_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \le i \le m+n)$ are unknown complex vector functions.

The above equation for $a = 1$ was solved in [1] under the assumption that the functions and variables are real. But the argument given there is valid only if the greatest common divisor of m and n is 1. Also, one special general case is solved in [2]. The theorems of [2] concerning the cases $m \neq n$ should be modified to give the general continuous solutions.

1. **Main Results**

Now we will give the following results.

Theorem 1. *If $a = 1$, $(m, n) = 1$ and $m + n > 2$, then the general continuous solution of the functional equation (1) is*

$$f_i(\mathbf{U}, \mathbf{V}) = F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_i(\mathbf{U}+\mathbf{V}) \tag{2}$$

$$(1 \le i \le m+n),$$

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so that

$$\sum_{i=1}^{n+m} G_i(\mathbf{U}) = -m[F_1(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U})\mathrm{Im}\,\mathbf{U}],$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n-1)$ *are arbitrary continuous complex vector functions.*

Proof. We accept the convention to reduce the indices $\mathrm{mod}(m+n)$. If we set

$$\mathbf{S} = \sum_{i=1}^{m+n} \mathbf{Z}_i,$$

$$\mathbf{T}_i = \mathbf{Z}_i + \mathbf{Z}_{i+1} + \cdots + \mathbf{Z}_{i+m-1} - \frac{m\mathbf{S}}{m+n} \qquad (1 \le i \le m+n-1), \tag{3}$$

the vectors $\mathbf{T}_i$ $(1 \le i \le m+n-1)$ and $\mathbf{S}$ are independent since $(m, n) = 1$. The equation (1) becomes

$$\sum_{i=1}^{m+n-1} f_i\Big(\mathbf{T}_i + \frac{m\mathbf{S}}{m+n},\ \frac{n\mathbf{S}}{m+n} - \mathbf{T}_i\Big) \tag{4}$$

$$+f_{m+n}\Big(-\mathbf{T}_1 - \mathbf{T}_2 - \cdots - \mathbf{T}_{m+n-1} + \frac{m\mathbf{S}}{m+n},\ \frac{n\mathbf{S}}{m+n} + \mathbf{T}_1 + \mathbf{T}_2 + \cdots + \mathbf{T}_{m+n-1}\Big) = \mathbf{O}.$$

We introduce the new notations

$$f_i\Big(\mathbf{U} + \frac{m\mathbf{S}}{m+n},\ \frac{n\mathbf{S}}{m+n} - \mathbf{U}\Big) = g_i(\mathbf{U}, \mathbf{S}) \quad (1 \le i \le m+n),$$

i.e.,

$$f_i(\mathbf{U}, \mathbf{V}) = g_i\Big(\frac{n\mathbf{U} - m\mathbf{V}}{m+n},\ \mathbf{U} + \mathbf{V}\Big) \quad (1 \le i \le m+n). \tag{5}$$

The equation (4) is transformed into

$$\sum_{i=1}^{m+n-1} g_i(\mathbf{T}_i, \mathbf{S}) + g_{m+n}(-\mathbf{T}_1 - \mathbf{T}_2 - \cdots - \mathbf{T}_{m+n-1},\ \mathbf{S}) = \mathbf{O}. \tag{6}$$

By the substitution $\mathbf{T}_1 = \mathbf{T}_2 = \cdots = \mathbf{T}_{r-1} = \mathbf{T}_{r+1} = \cdots = \mathbf{T}_{m+n-1} = \mathbf{O}$, we obtain

$$g_r(\mathbf{T}_r, \mathbf{S}) = -g_{m+n}(-\mathbf{T}_r, \mathbf{S}) - H_r(\mathbf{S}) \quad (1 \le r \le m+n-1). \tag{7}$$

Putting (7) into (6), we get

$$g_{m+n}(-\mathbf{T}_1 - \mathbf{T}_2 - \cdots - \mathbf{T}_{m+n-1},\ \mathbf{S}) = \sum_{i=1}^{m+n-1} g_{m+n}(-\mathbf{T}_i, \mathbf{S}) + \sum_{i=1}^{m+n-1} H_i(\mathbf{S}). \tag{8}$$

We conclude that the function

$$K(\mathbf{U}, \mathbf{S}) = g_{m+n}(\mathbf{U}, \mathbf{S}) + \frac{1}{m+n-2} \sum_{i=1}^{m+n-1} H_i(\mathbf{S}) \tag{9}$$

satisfies the functional equation

$$K(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n-1},\ \mathbf{S}) = \sum_{i=1}^{m+n-1} K(\mathbf{Z}_i, \mathbf{S}). \tag{10}$$

Using the continuity of K, from (10) we deduce that for fixed $\mathbf{S}$

$$K(\mathbf{U},\mathbf{S}) = c_1 \mathrm{Re}\,\mathbf{U} + c_2 \mathrm{Im}\,\mathbf{U},$$

where $\mathrm{Re}\,\mathbf{U}$ resp. $\mathrm{Im}\,\mathbf{U}$ denotes the real resp. imaginary part of $\mathbf{U}$. The mappings $c_1, c_2 \in \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ may depend upon $\mathbf{S}$. Hence,

$$K(\mathbf{U},\mathbf{V}) = F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{U}, \tag{11}$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ are continuous functions.

From (9), (11) and (7) we obtain

$$\begin{aligned} g_{m+n}(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + f_2(\mathbf{V})\mathrm{Im}\,\mathbf{U} - \frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{V}),\\ g_r(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{U} - H_r(\mathbf{V}) \qquad (12)\\ &+ \frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{V}) \quad (1 \le r \le m+n-1). \end{aligned}$$

From (5) and (12) we deduce that

$$\begin{aligned} f_r(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\Big(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\Big) + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\Big(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\Big)\\ &-H_r(\mathbf{U}+\mathbf{V}) + \frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}) \quad (1 \le r \le m+n-1),\\ f_{m+n}(\mathbf{U}+\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\Big(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\Big) + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\Big(\frac{n\mathbf{U}-m\mathbf{V}}{m+n}\Big)\\ &-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}). \qquad (13) \end{aligned}$$

By denoting

$$\begin{aligned} &-F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\Big[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\Big] - F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\Big[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\Big]\\ &+\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}) - H_r(\mathbf{U}+\mathbf{V}) = G_r(\mathbf{U}+\mathbf{V}) \quad (1 \le r \le m+n-1),\\ &-F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\Big[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\Big] - F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\Big[\frac{m(\mathbf{U}+\mathbf{V})}{m+n}\Big]\\ &-\frac{1}{m+n-2}\sum_{i=1}^{m+n-1} H_i(\mathbf{U}+\mathbf{V}) = G_{m+n}(\mathbf{U}+\mathbf{V}), \end{aligned}$$

from (13) we get (2).

The converse can be established by a straightforward verification. □

Example 1. The general continuous solution of the functional equation

$$f_1(\mathbf{Z}_1+\mathbf{Z}_2,\ \mathbf{Z}_3) + f_2(\mathbf{Z}_2+\mathbf{Z}_3,\ \mathbf{Z}_1) + f_3(\mathbf{Z}_3+\mathbf{Z}_1,\ \mathbf{Z}_2) = \mathbf{O}$$

is given by

$$\begin{aligned} f_1(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_1(\mathbf{U}+\mathbf{V}),\\ f_2(\mathbf{U},\mathbf{V}) &= F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_2(\mathbf{U}+\mathbf{V}), \end{aligned}$$

$f_3(\mathbf{U},\mathbf{V}) = -F_1(\mathbf{U}+\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2\mathbf{V}) - F_2(\mathbf{U}+\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2\mathbf{V}) - G_1(\mathbf{U}+\mathbf{V}) - G_2(\mathbf{U}+\mathbf{V})$,
where $F_1, F_2 : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ and $G_1, G_2 : \mathcal{V} \to \mathcal{V}'$ are arbitrary continuous complex vector functions.

Corollary. *The general continuous solution of the vector functional equation*

$$\sum_{i=1}^{m+n} g_i(\mathbf{Z}_i + \cdots + \mathbf{Z}_{i+m-1},\ \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n}) = \mathbf{O}$$

if $(m,n) = 1$ *and* $m+n > 2$ *is given by*

$$g_i(\mathbf{U},\mathbf{V}) = F_1(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_i(\mathbf{V}) \quad (1 \le i \le m+n),$$

$$\sum_{i=1}^{m+n} G_i(\mathbf{V}) = -m[F_1(\mathbf{V})\mathrm{Re}\,\mathbf{V} + F_2(\mathbf{V})\mathrm{Im}\,\mathbf{V}],$$

where $F_1, F_2 : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$, $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n-1)$ *are arbitrary continuous complex vector functions.*

Proof. Put $f_i(\mathbf{U},\mathbf{V}) = g_i(\mathbf{U},\mathbf{U}+\mathbf{V})$ in Theorem 1. □

Theorem 2. *The general continuous solution of the complex vector functional equation (1) if* $a = 1$, $(m,n) = d > 1$, $m/d = p$, $n/d = q$ *and* $p+q > 2$ *is given by*

$$\begin{gathered} f_{id+j}(\mathbf{U},\mathbf{V}) = F_{1j}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{2j}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{ij}(\mathbf{U}+\mathbf{V}) \\ (0 \le i \le p+q-1, \quad 1 \le j \le d), \end{gathered}$$

$$\sum_{i=0}^{p+q-1} G_{ij}(\mathbf{U}) = H_j(\mathbf{U}) - p[F_{1j}(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_{2j}(\mathbf{U})\mathrm{Im}\,\mathbf{U}] \quad (1 \le j \le d), \tag{14}$$

$$\sum_{j=1}^{d} H_j(\mathbf{U}) = \mathbf{O},$$

where

$$\begin{gathered} F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1,2;\ 1 \le j \le d), \\ H_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1), \\ G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2;\ 1 \le j \le d) \end{gathered}$$

are arbitrary continuous complex vector functions.

Proof. We set

$$f_i(\mathbf{U},\mathbf{V}) = g_i(\mathbf{U},\mathbf{U}+\mathbf{V}) \quad (1 \le i \le m+n) \tag{15}$$

and we obtain

$$\sum_{i=1}^{m+n} g_i(\mathbf{Z}_i + \mathbf{Z}_{i+1} + \cdots + \mathbf{Z}_{i+m-1},\ \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n}) = \mathbf{O}. \tag{16}$$

Let us introduce the new vectors

$$\mathbf{V}_i = \mathbf{Z}_i + \mathbf{Z}_{i+1} + \cdots + \mathbf{Z}_{i+d-1} \quad (1 \le i \le m+n) \quad \text{so that} \quad \mathbf{V}_{i+m+n} = \mathbf{V}_i \tag{17}$$

and

$$\mathbf{W} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_{m+n}. \tag{18}$$

They are not independent because

$$\sum_{i=0}^{p+q-1} \mathbf{V}_{id+j} = \mathbf{W} \quad (1 \le j \le d). \tag{19}$$

The vectors $\mathbf{V}_i$ $(1 \le i \le m+n-d)$ and $\mathbf{W}$ are independent because the rank of the matrix of linear forms determining them is $m+n-d+1$, which is easy to verify. In the sequel we will use all vectors (17) and (18) but we must have always in mind that (19) holds. The equation (16) becomes

$$\sum_{i=1}^{m+n} g_i(\mathbf{V}_i + \mathbf{V}_{i+d} + \cdots + \mathbf{V}_{i+(p-1)d},\ \mathbf{W}) = \mathbf{O}.$$

It can be written in the following form

$$\sum_{j=1}^{d} \sum_{i=0}^{p+q-1} g_{id+j}(\mathbf{V}_{id+j} + \mathbf{V}_{(i+1)d+j} + \cdots + \mathbf{V}_{(i+p-1)d+j},\ \mathbf{W}) = \mathbf{O}.$$

If we set here

$$\mathbf{V}_{id+j} = \mathbf{O} \quad (0 \le i \le p+q-2;\ j = 1,2,\dots,r-1,r+1,\dots,d),$$
$$\mathbf{V}_{(p+q-1)d+j} = \mathbf{W} \quad (j = 1,2,\dots,r-1,r+1,\dots,d),$$

we get

$$\sum_{i=0}^{p+q-1} g_{id+r}(\mathbf{V}_{id+r} + \mathbf{V}_{(i+1)d+r} + \cdots + \mathbf{V}_{(i+p-1)d+r},\ \mathbf{W}) - \frac{H_r(\mathbf{W})}{p+q} = \mathbf{O} \quad (1 \le r \le d)$$

and

$$\sum_{r=1}^{d} H_r(\mathbf{W}) = \mathbf{O}.$$

By using the corollary of Theorem 1 we get

$$g_{id+r}(\mathbf{U},\mathbf{V}) = F_{1r}(\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{2r}(\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{ir}(\mathbf{V}) \quad (0 \le i \le p+q-1;\ 1 \le r \le d),$$
$$\sum_{i=0}^{p+q-1} G_{ir}(\mathbf{V}) = H_r(\mathbf{V}) - p[F_{1r}(\mathbf{V})\mathrm{Re}\,\mathbf{V} + F_{2r}(\mathbf{V})\mathrm{Im}\,\mathbf{V}] \quad (1 \le r \le d),$$

where

$$F_{ir} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1,2;\ 1 \le r \le d),$$
$$G_{ir} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2;\ 1 \le r \le d),$$
$$H_r : \mathcal{V} \to \mathcal{V}' \quad (1 \le r \le d-1)$$

are arbitrary continuous complex vector functions. By application of (15) these formulas give (14).

It is easy to prove that the functions $f_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \le i \le m+n)$ defined by (15) satisfy the complex vector functional equations (1). $\square$

Example 2. The general continuous solution of the functional equation

$$f_1(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4,\ \mathbf{Z}_5 + \mathbf{Z}_6) + f_2(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5,\ \mathbf{Z}_6 + \mathbf{Z}_1)$$
$$+ f_3(\mathbf{Z}_3 + \mathbf{Z}_4 + \mathbf{Z}_5 + \mathbf{Z}_6,\ \mathbf{Z}_1 + \mathbf{Z}_2) + f_4(\mathbf{Z}_4 + \mathbf{Z}_5 + \mathbf{Z}_6 + \mathbf{Z}_1,\ \mathbf{Z}_2 + \mathbf{Z}_3)$$
$$+ f_5(\mathbf{Z}_5 + \mathbf{Z}_6 + \mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3 + \mathbf{Z}_4) + f_6(\mathbf{Z}_6 + \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_4 + \mathbf{Z}_5) = \mathbf{O}$$

is given by

$$f_1(\mathbf{U},\mathbf{V}) = F_{11}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{21}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{01}(\mathbf{U}+\mathbf{V}),$$
$$f_2(\mathbf{U},\mathbf{V}) = F_{12}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{22}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{02}(\mathbf{U}+\mathbf{V}),$$
$$f_3(\mathbf{U},\mathbf{V}) = F_{11}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{21}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{11}(\mathbf{U}+\mathbf{V}),$$

$$f_4(\mathbf{U},\mathbf{V}) = F_{12}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,\mathbf{U} + F_{22}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,\mathbf{U} + G_{12}(\mathbf{U}+\mathbf{V}),$$
$$f_5(\mathbf{U},\mathbf{V}) = -F_{11}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2\mathbf{V}) - F_{21}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2\mathbf{V})$$
$$+H_1(\mathbf{U}+\mathbf{V}) - G_{01}(\mathbf{U}+\mathbf{V}) - G_{11}(\mathbf{U}+\mathbf{V}),$$
$$f_6(\mathbf{U},\mathbf{V}) = -F_{12}(\mathbf{U}+\mathbf{V})\mathrm{Re}\,(\mathbf{U}+2\mathbf{V}) - F_{22}(\mathbf{U}+\mathbf{V})\mathrm{Im}\,(\mathbf{U}+2\mathbf{V})$$
$$-H_1(\mathbf{U}+\mathbf{V}) - G_{01}(\mathbf{U}+\mathbf{V}) - G_{12}(\mathbf{U}+\mathbf{V}),$$

where

$$F_{ij}:\mathcal{V}\to\mathcal{L}(\mathcal{V}^0,\mathcal{V}') \quad (i=1,2),$$
$$G_{ij}:\mathcal{V}\to\mathcal{V}' \quad (i=0,1;\ j=1,2),$$
$$H_1:\mathcal{V}\to\mathcal{V}'$$

are arbitrary continuous complex vector functions.

Theorem 3. *The most general solution of (1) if $a=1$ and $m=n$ is*

$$f_i(\mathbf{U},\mathbf{V}) \quad (1\le i\le m) \quad \textit{are} \quad \textit{arbitrary},$$
$$f_{m+i}(\mathbf{U},\mathbf{V}) = H_i(\mathbf{U}+\mathbf{V}) - f_i(\mathbf{V},\mathbf{U}) \quad (1\le i\le m), \tag{20}$$
$$\sum_{i=1}^{m} H_i(\mathbf{U}) = \mathbf{O},$$

where $H_i:\mathcal{V}\to\mathcal{V}'$ $(1\le i\le m-1)$ are arbitrary functions.

Proof. Put $f_i(\mathbf{U},\mathbf{V}) = G_i(\mathbf{U},\mathbf{U}+\mathbf{V})$. □

Example 3. The most general solution of the equation

$$f_1(\mathbf{Z}_1+\mathbf{Z}_2,\ \mathbf{Z}_3+\mathbf{Z}_4) + f_2(\mathbf{Z}_2+\mathbf{Z}_3,\ \mathbf{Z}_4+\mathbf{Z}_1)$$
$$+f_3(\mathbf{Z}_3+\mathbf{Z}_4,\ \mathbf{Z}_1+\mathbf{Z}_2) + f_4(\mathbf{Z}_4+\mathbf{Z}_1,\ \mathbf{Z}_2+\mathbf{Z}_3) = \mathbf{O}$$

is

$$f_1(\mathbf{U},\mathbf{V}),\ f_2(\mathbf{U},\mathbf{V}) \quad \textit{are} \quad \textit{arbitrary},$$
$$f_3(\mathbf{U},\mathbf{V}) = H_1(\mathbf{U}+\mathbf{V}) - f_1(\mathbf{V},\mathbf{U}),$$
$$f_4(\mathbf{U},\mathbf{V}) = -H_1(\mathbf{U}+\mathbf{V}) - f_2(\mathbf{V},\mathbf{U}),$$

where $H_1:\mathcal{V}\to\mathcal{V}'$ is an arbitrary function.

Theorem 4. *If $a^{m+n}\ne 1$ and $m\ne n$, the general solution of the functional equation (1) is given by*

$$f_i(\mathbf{U},\mathbf{V}) = F_i(\mathbf{U}+a^m\mathbf{V}) - F_{i+n}(a^n\mathbf{U}+\mathbf{V}) + A_i \quad (1\le i\le m+n), \tag{21}$$

where $F_i:\mathcal{V}\to\mathcal{V}'$ $(1\le i\le m+n)$ are arbitrary complex vector functions, and A_i are arbitrary constant complex vectors such that $\sum\limits_{i=1}^{m+n} A_i = \mathbf{O}$.

Proof. If we introduce new functions g_i by the equation

$$f_i(\mathbf{U},\mathbf{V}) = g_i(\mathbf{U}+a^m\mathbf{V},\ a^n\mathbf{U}+\mathbf{V}) \quad (1\le i\le m+n), \tag{22}$$

then equation (1) becomes

$$\sum_{i=1}^{m+n} g_i\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n-1-j}\mathbf{Z}_{m+i+j},$$
$$\sum_{j=0}^{m-1} a^{m+n-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{m+i+j}\Big) = \mathbf{O},$$

i.e.,

$$\sum_{i=1}^{m+n} g_i\Big(\sum_{j=0}^{m+n-1} a^j \mathbf{Z}_{m+i-1-j},\ \sum_{j=0}^{m+n-1} a^j \mathbf{Z}_{i-1-j}\Big) = \mathbf{O}. \tag{23}$$

Since $a^{m+n} \neq 1$, this transformation is possible. Also we may introduce new vectors $\mathbf{V}_i$ by

$$\mathbf{V}_i = \sum_{j=0}^{m+n-1} a^j \mathbf{Z}_{m+i-1-j} \quad (1 \le i \le m+n)$$

but the equation (23) takes the form

$$\sum_{i=0}^{m+n} g_i(\mathbf{V}_i, \mathbf{V}_{i+n}) = \mathbf{O}. \tag{24}$$

By putting $\mathbf{V}_j = \mathbf{O}$ $(j = 1, 2, \dots, i-1, i+1, \dots, i+n-1, i+n+1, \dots, m+n)$ we obtain

$$g_i(\mathbf{V}_i, \mathbf{V}_{i+n}) = F_i(\mathbf{V}_i) + G_i(\mathbf{V}_{i+n}) \quad (1 \le i \le m+n). \tag{25}$$

On the basis of the expression (25), the equation (24) becomes

$$\sum_{i=1}^{m+n} [F_i(\mathbf{V}_i) + G_i(\mathbf{V}_{i+n})] = \mathbf{O},$$

or

$$\sum_{i=1}^{m+n} [F_i(\mathbf{V}_i) + G_{m+i}(\mathbf{V}_i)] = \mathbf{O}. \tag{26}$$

From (26) it follows that

$$G_{i+m}(\mathbf{V}_i) = -F_i(\mathbf{V}_i) + A_i \quad (1 \le i \le m+n), \tag{27}$$

where A_i are arbitrary constant complex vectors with the property

$$\sum_{i=1}^{m+n} A_i = \mathbf{O}.$$

On the basis of the expression (27), the equality (25) has the form

$$g_i(\mathbf{U}, \mathbf{V}) = F_i(\mathbf{U}) + F_{i+n}(\mathbf{V}) + A_i \quad (1 \le i \le m+n), \tag{28}$$

where $\sum\limits_{i=1}^{m+n} A_i = \mathbf{O}$.

On the basis of the equalities (28) and (22), we obtain (21). □

Example 4. If $a^3 \neq 1$, the general solution of the functional equation

$$f_1(a^2\mathbf{Z}_1 + a\mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_4) + f_2(a^2\mathbf{Z}_2 + a\mathbf{Z}_3 + \mathbf{Z}_4,\ \mathbf{Z}_1)$$
$$+ f_3(a^2\mathbf{Z}_3 + a\mathbf{Z}_4 + \mathbf{Z}_1,\ \mathbf{Z}_2) + f_4(a^2\mathbf{Z}_4 + a\mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f_1(\mathbf{U}, \mathbf{V}) = F_1(\mathbf{U} + a^3\mathbf{V}) - F_2(a\mathbf{U} + \mathbf{V}) + A_1,$$
$$f_2(\mathbf{U}, \mathbf{V}) = F_2(\mathbf{U} + a^3\mathbf{V}) - F_3(a\mathbf{U} + \mathbf{V}) + A_2,$$
$$f_3(\mathbf{U}, \mathbf{V}) = F_3(\mathbf{U} + a^3\mathbf{V}) - F_4(a\mathbf{U} + \mathbf{V}) + A_3,$$
$$f_4(\mathbf{U}, \mathbf{V}) = F_4(\mathbf{U} + a^3\mathbf{V}) - F_1(a\mathbf{U} + \mathbf{V}) - A_1 - A_2 - A_3,$$

where $F_i : \mathcal{V} \to \mathcal{V}'$ $(i = 1, 2, 3, 4)$ are arbitrary complex vector functions, and A_i $(i = 1, 2, 3)$ are arbitrary constant complex vectors.

Theorem 5. *If $a^{m+n} \neq 1$ and $m = n$, the most general solution of the functional equation (1) is*

$$f_{i+m}(\mathbf{U}, \mathbf{V}) = -f_i(\mathbf{V}, \mathbf{U}) + A_i \quad (1 \leq i \leq m), \tag{29}$$

where $f_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \leq i \leq m)$ and A_i $(1 \leq i \leq m)$ are arbitrary complex constant vectors such that $\sum_{i=1}^{m} A_i = \mathbf{O}$.

Proof. By the transformations which were exhibited in the proof of the previous theorem we may bring the equation (1) to the form (24).

For $\mathbf{V}_j = \mathbf{O}$ $(j = 1, 2, \ldots, i-1, i+1, \ldots, i+m-1, i+m+1, \ldots, 2m)$ the equation (24) becomes

$$g_i(\mathbf{V}_i, \mathbf{V}_{i+m}) + g_{i+m}(\mathbf{V}_{i+m}, \mathbf{V}_i) = A_i \quad (1 \leq i \leq m), \tag{30}$$

where A_i $(1 \leq i \leq m)$ are arbitrary complex constant vectors. By substituting (30) into (1), we obtain that it must hold

$$\sum_{i=1}^{m} A_i = \mathbf{O}.$$

On the basis of this equality and (30), we obtain (29). □

Example 5. If $a^4 \neq 1$, the most general solution of the functional equation

$$f_1(a\mathbf{Z}_1 + \mathbf{Z}_2,\ a\mathbf{Z}_3 + \mathbf{Z}_4) + f_2(a\mathbf{Z}_2 + \mathbf{Z}_3,\ a\mathbf{Z}_4 + \mathbf{Z}_1)$$
$$+ f_3(a\mathbf{Z}_3 + \mathbf{Z}_4,\ a\mathbf{Z}_1 + \mathbf{Z}_2) + f_4(a\mathbf{Z}_4 + \mathbf{Z}_1,\ a\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f_i(\mathbf{U}, \mathbf{V}) \quad (i = 1, 2) \quad \textit{are} \quad \textit{arbitrary},$$
$$f_3(\mathbf{U}, \mathbf{V}) = -f_1(\mathbf{U}, \mathbf{V}) + A,$$
$$f_4(\mathbf{U}, \mathbf{V}) = -f_1(\mathbf{U}, \mathbf{V}) - A,$$

where A is an arbitrary complex constant vector.

If $a^{m+n} = 1$, then the functional equation (1) may be transformed in the following way.

We introduce new vectors by the equality

$$\mathbf{V}_i = a^{1-i}\mathbf{Z}_i, \quad \text{i.e.,} \quad \mathbf{Z}_i = a^{i-1}\mathbf{V}_i \quad (1 \leq i \leq m+n).$$

Then the equation (1) becomes

$$\sum_{i=1}^{m+n} f_i\Big(a^{m-2+i} \sum_{j=0}^{m-1} \mathbf{V}_{i+j},\ a^{m+n-2+i} \sum_{j=0}^{n-1} \mathbf{V}_{m+i+j}\Big) = \mathbf{O}. \tag{31}$$

Now, if we put

$$g_i(\mathbf{U}, \mathbf{V}) = f_i(a^{m-2+i}\mathbf{U},\ a^{m+n-2+i}\mathbf{V}) \quad (1 \leq i \leq m+n),$$

i.e.,

$$f_i(\mathbf{U}, \mathbf{V}) = g_i(a^{n+2-i}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \leq i \leq m+n), \tag{32}$$

the functional equation (31) takes the form

$$\sum_{i=1}^{m+n} g_i\Big(\sum_{j=0}^{m-1} \mathbf{V}_{i+j},\ \sum_{j=0}^{n-1} \mathbf{V}_{m+i+j}\Big) = \mathbf{O}. \tag{33}$$

The equation (33) is just the equation (1) for $a = 1$.

Theorem 6. *If $a^{m+n} = 1,\ (m, n) = 1$ and $m + n > 2$, then the general continuous solution of the functional equation (1) is given by*

$$f_i(\mathbf{U}, \mathbf{V}) = F_1(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Re}\,(a^{n+2-i}\mathbf{U}) \tag{34}$$

$$+F_2(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Im}\,(a^{n+2-i}\mathbf{U}) + G_i(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})$$

$(1 \le i \le m + n)$, so that

$$\sum_{i=1}^{m+n} G_i(\mathbf{U}) = -m[F_1(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_2(\mathbf{U})\mathrm{Im}\,\mathbf{U}], \tag{35}$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n-1)$ are arbitrary continuous complex vector functions.

Proof. The proof immediately follows from (33), (32) and Theorem 1. □

Theorem 7. *If $a^{m+n} = 1,\ (m, n) = d > 1,\ m/d = p,\ n/d = q$ and $p + q > 2$, then the general continuous solution of the functional equation (1) is*

$$f_{id+j}(\mathbf{U}, \mathbf{V}) = F_{1j}(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Re}\,(a^{n+2-i}\mathbf{U}) \tag{36}$$

$$+F_{2j}(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Im}\,(a^{n+2-i}\mathbf{U}) + G_{ij}(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})$$

$$(0 \le i \le p + q - 1; \quad 1 \le j \le d)$$

so that

$$\sum_{i=0}^{p+q-1} G_{ij}(\mathbf{U}) = H_j(\mathbf{U}) - p[F_{1j}(\mathbf{U})\mathrm{Re}\,\mathbf{U} + F_{2j}(\mathbf{U})\mathrm{Im}\,\mathbf{U}] \quad (1 \le j \le d), \tag{37}$$

$$\sum_{j=1}^{d} H_j(\mathbf{U}) = \mathbf{O}, \tag{38}$$

where

$$F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2; \quad 1 \le j \le d),$$

$$G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p + q - 2; \quad 1 \le j \le d),$$

$$H_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d - 1)$$

are arbitrary continuous complex vector functions.

Proof. On the basis of the expressions (33), (32) and Theorem 2 we derive the proof of the theorem. □

Theorem 8. *If $a^{m+n} = 1$ and $m = n$, then the most general solution of the functional equation (1) is given by*

$$\begin{aligned} f_i(\mathbf{U}, \mathbf{V}) & \quad (1 \le i \le m) \quad \text{are} \quad \text{arbitrary}, \\ f_{m+i}(\mathbf{U}, \mathbf{V}) &= H_i(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V}) \\ &- f_i(a^{n+2-i}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \le i \le m), \end{aligned} \tag{39}$$

where $H_i : \mathcal{V} \to \mathcal{V}'$ are arbitrary complex vector functions such that $\sum\limits_{i=1}^{m} H_i(\mathbf{U}) = \mathbf{O}$.

Proof. The proof immediately follows from (33), (32) and Theorem 3. □

2. **A Special Functional Equation**

Now, we will solve the following functional equation

$$\sum_{i=1}^{m+n} f\Big(\sum_{j=0}^{m-1} a^{m-1-j}\mathbf{Z}_{i+j},\ \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}\Big) = \mathbf{O}, \tag{40}$$

which is obtained as a special case of the equation (1) for $f_i = f$ $(1 \le i \le m+n)$.

Theorem 9. *If $a^{m+n} \neq 1$, then the most general solution of the complex vector functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \begin{cases} F(\mathbf{U}+a^m\mathbf{V}) - F(a^n\mathbf{U}+\mathbf{V}) & (m \neq n), \\ G(\mathbf{U}+a^m\mathbf{V},\ a^m\mathbf{U}+\mathbf{V}) - G(a^m\mathbf{U}+\mathbf{V},\ \mathbf{U}+a^m\mathbf{V}) & (m = n), \end{cases} \tag{41}$$

where $F : \mathcal{V} \to \mathcal{V}'$, $G : \mathcal{V}^2 \to \mathcal{V}'$ are arbitrary complex vector functions.

Proof. We set

$$f(\mathbf{U},\mathbf{V}) = g(\mathbf{U}+a^m\mathbf{V},\ a^n\mathbf{U}+\mathbf{V}) \tag{42}$$

into (40) and deduce that

$$\sum_{i=1}^{m+n} g\Big(\sum_{j=0}^{m+1} a^{m-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{m+n-1}\mathbf{Z}_{i+m+j},$$

$$\sum_{j=0}^{m-1} a^{m+n-1-j}\mathbf{Z}_{i+j} + \sum_{j=0}^{n-1} a^{n-1-j}\mathbf{Z}_{i+m+j}\Big) = \mathbf{O},$$

i.e.,

$$\sum_{i=1}^{m+n} g\Big(\sum_{j=0}^{m+n-1} a^{j}\mathbf{Z}_{i+m-1-j},\ \sum_{j=0}^{m+n-1} a^{j}\mathbf{Z}_{i-1-j}\Big) = \mathbf{O}. \tag{43}$$

This transformation of the equation (40) is possible since $a^{m+n} \neq 1$.

Now we introduce new vectors

$$\mathbf{V}_i = \sum_{j=0}^{m+n+1} a^j\mathbf{Z}_{i-1-j} \quad (1 \le i \le m+n). \tag{44}$$

The linear forms (44) are independent since their determinant is $(a^{m+n}-1)^{m+n-1}$.

Making use of these notations, the equation (43) becomes

$$\sum_{i=1}^{m+n} g(\mathbf{V}_i, \mathbf{V}_{i+n}) = \mathbf{O}. \tag{45}$$

If $m \neq n$, we set $\mathbf{V}_1 = \mathbf{V}_2 = \cdots = \mathbf{V}_{m-1} = \mathbf{V}_{m+1} = \mathbf{V}_{m+2} = \cdots = \mathbf{V}_{m+n-1} = \mathbf{O}$ and we get

$$g(\mathbf{U},\mathbf{V}) = F(\mathbf{U}) + F_1(\mathbf{V}). \tag{46}$$

We substitute g from (46) into (45) and obtain

$$\sum_{i=1}^{m+n} [F(\mathbf{V}_i) + F_1(\mathbf{V}_i)] = \mathbf{O},$$

which implies that $F_1(\mathbf{V}_i) = -F(\mathbf{V}_i)$. Hence,

$$g(\mathbf{U},\mathbf{V}) = F(\mathbf{U}) - F(\mathbf{V}). \tag{47}$$

If $m = n$, the equation (43) yields

$$g(\mathbf{U},\mathbf{V}) + g(\mathbf{V},\mathbf{U}) = \mathbf{O},$$

i.e.,

$$g(\mathbf{U},\mathbf{V}) = G(\mathbf{U},\mathbf{V}) - G(\mathbf{V},\mathbf{U}). \tag{48}$$

From (42), (47) and (48) we conclude that (41) holds. It is easy to verify that (41) satisfies (40). □

Example 6. If $a^3 \neq 1$, then the most general solution of the functional equation

$$f(a\mathbf{Z}_1+\mathbf{Z}_2,\ \mathbf{Z}_3) + f(a\mathbf{Z}_2+\mathbf{Z}_3,\ \mathbf{Z}_1) + f(a\mathbf{Z}_3+\mathbf{Z}_1,\ \mathbf{Z}_2) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F(\mathbf{U}+a^2\mathbf{V}) - F(a\mathbf{U}+\mathbf{V}),$$

where $F:\mathcal{V}\to\mathcal{V}'$ is an arbitrary complex vector function.

Example 7. If $a^4 \neq 1$, the most general solution of the functional equation

$$f(a\mathbf{Z}_1+\mathbf{Z}_2,\ a\mathbf{Z}_3+\mathbf{Z}_4) + f(a\mathbf{Z}_2+\mathbf{Z}_3,\ a\mathbf{Z}_4+\mathbf{Z}_1)$$
$$+f(a\mathbf{Z}_3+\mathbf{Z}_4,\ a\mathbf{Z}_1+\mathbf{Z}_2) + f(a\mathbf{Z}_4+\mathbf{Z}_1,\ a\mathbf{Z}_2+\mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = G(\mathbf{U}+a^2\mathbf{V},\ a^2\mathbf{U}+\mathbf{V}) - G(a^2\mathbf{U}+\mathbf{V},\ \mathbf{U}+a^2\mathbf{V}),$$

where $G:\mathcal{V}^2\to\mathcal{V}'$ is an arbitrary complex vector function.

Theorem 10. *If $a^{m+n}=1$, $(m,n)=1$ and $m+n>2$, then the general continuous solution of the functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \sum_{i=1}^{m+n}[F_1(a^i\mathbf{U}+a^{i+m}\mathbf{V})\mathrm{Re}\,(a^i\mathbf{U}) + F_2(a^i\mathbf{U}+a^{i+m}\mathbf{V})\mathrm{Im}\,(a^i\mathbf{U})] \tag{49}$$
$$+\sum_{i=1}^{m+n-1}[G_i(a^i\mathbf{U}+a^{i+m}\mathbf{V}) - G_i(a^i\mathbf{U}+a^m\mathbf{V})]$$
$$-m[F_1(\mathbf{U}+a^m\mathbf{V})\mathrm{Re}\,(\mathbf{U}+a^m\mathbf{V}) + F_2(\mathbf{U}+a^m\mathbf{V})\mathrm{Im}\,(\mathbf{U}+a^m\mathbf{V})],$$

where $F_i:\mathcal{V}\to\mathcal{L}(\mathcal{V}^0,\mathcal{V}')$ $(i=1,2)$ and $G_i:\mathcal{V}\to\mathcal{V}'$ $(1\le i\le m+n-1)$ are arbitrary complex vector functions.

Proof. Let us put $\mathbf{Z}_i = a^{i-1}\mathbf{T}_i$ $(1\le i\le m+n)$. The equation (40) becomes

$$\sum_{i=1}^{m+n} f\Big(a^{m+i-2}\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ a^{m+n-2+i}\sum_{j=0}^{n-1}\mathbf{T}_{m+i-j}\Big) = \mathbf{O}. \tag{50}$$

Now we make the substitutions

$$f(a^{m+i-2}\mathbf{U},\ a^{m+n-2+i}\mathbf{V}) = f_i(\mathbf{U},\mathbf{V}) \quad (1\le i\le m+n),$$

i.e.,

$$f(\mathbf{U},\mathbf{V}) = f_i(a^{n-i+2}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1\le i\le m+n), \tag{51}$$

and we obtain

$$\sum_{i=1}^{m+n} f_i\Big(\sum_{j=0}^{m-1}\mathbf{T}_{i+j},\ \sum_{j=0}^{n-1}\mathbf{T}_{m+i+j}\Big) = \mathbf{O}. \tag{52}$$

The equation (52) is just the equation (1) for $a = 1$, and its solution is determined by Theorem 1.

By an application of Theorem 1, and by (51) we get

$$f_i(\mathbf{U},\mathbf{V}) = P_1(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Re}\,(a^{n+2-i}\mathbf{U}) \quad (53)$$
$$+P_2(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})\mathrm{Im}\,(a^{n+2-i}\mathbf{U}) + Q_i(a^{n+2-i}\mathbf{U} + a^{m+n+2-i}\mathbf{V})$$

$(1 \le i \le m+n)$, so that

$$\sum_{i=1}^{m+n} Q_i(\mathbf{U}) = -m[P_1(\mathbf{U})\mathrm{Re}\,\mathbf{U} + P_2(\mathbf{U})\mathrm{Im}\,\mathbf{U}],$$

where $P_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $Q_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m+n)$ are continuous complex vector functions. By addition of all equations (53) and putting

$$P_1(\mathbf{U}) = (m+n)F_1(\mathbf{U}), \quad P_2(\mathbf{U}) = (m+n)F_2(\mathbf{U}),$$
$$Q_i(\mathbf{U}) = (m+n)G_{n+2-i}(\mathbf{U}) \quad (i = 1, 2, \dots, m+n)$$

we obtain (49). □

Example 8. If $a^3 = 1$, the general continuous solution of the functional equation

$$f(a\mathbf{Z}_1 + \mathbf{Z}_2,\ \mathbf{Z}_3) + f(a\mathbf{Z}_2 + \mathbf{Z}_3,\ \mathbf{Z}_1) + f(a\mathbf{Z}_3 + \mathbf{Z}_1,\ \mathbf{Z}_2) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F_1(a\mathbf{U} + \mathbf{V})\mathrm{Re}\,(a\mathbf{U}) + F_2(a\mathbf{U} + \mathbf{V})\mathrm{Im}\,(a\mathbf{U})$$
$$+F_1(a^2\mathbf{U} + \mathbf{V})\mathrm{Re}\,(a^2\mathbf{U}) + F_2(a^2\mathbf{U} + \mathbf{V})\mathrm{Im}\,(a^2\mathbf{U})$$
$$-F_1(\mathbf{U} + a^2\mathbf{V})\mathrm{Re}\,(\mathbf{U} + 2a^2\mathbf{V}) - F_2(\mathbf{U} + a^2\mathbf{V})\mathrm{Im}\,(\mathbf{U} + 2a^2\mathbf{V})$$
$$+G_1(a\mathbf{U} + \mathbf{V}) - G_1(\mathbf{U} + a^2\mathbf{V}) + G_2(a^2\mathbf{U} + a\mathbf{V}) - G_2(\mathbf{U} + a^2\mathbf{V}),$$

where $F_i : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i = 1, 2)$ and $G_i : \mathcal{V} \to \mathcal{V}'$ $(i = 1, 2)$ are arbitrary complex vector functions.

Theorem 11. *If $a^{m+n} = 1$, $(m, n) = d > 1$, $m/d = p$, $n/d = q$ and $p + q > 2$, then the general continuous solution of the functional equation (40) is given by*

$$f(\mathbf{U},\mathbf{V}) = \sum_{j=-1}^{d-2}\sum_{i=0}^{p+q-1} [F_{1,j+2}(a^{n-id-j}\mathbf{U} + a^{-id-j}\mathbf{V})\mathrm{Re}\,(a^{n-id-j}\mathbf{U}) \quad (54)$$
$$+F_{2,j+2}(a^{n-id-j}\mathbf{U} + a^{-id-j}\mathbf{V})\mathrm{Im}\,(a^{n-id-j}\mathbf{U}) + G_{i,j+2}(a^{n-id-j}\mathbf{U} + a^{-id-j}\mathbf{V})],$$

so that

$$\sum_{i=0}^{p+q-1} G_{ij}(\mathbf{U}) = H_j(\mathbf{U}) - p[F_{1j}(\mathbf{U})\mathrm{Re}\,(\mathbf{U}) + F_{2j}(\mathbf{U})\mathrm{Im}\,(\mathbf{U})] \quad (1 \le j \le d)$$

and

$$\sum_{j=1}^{d} H_j(\mathbf{U}) = \mathbf{O},$$

where

$$F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2; \quad 1 \le j \le d),$$
$$G_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2; \quad 1 \le j \le d),$$
$$H_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1)$$

are arbitrary continuous complex vector functions.

Proof. We can start from equation (50). From (49) and (50) on the basis of Theorem 2 we get

$$f(\mathbf{U},\mathbf{V}) = P_{1j}(a^{n-id-j+2}\mathbf{U} + a^{m+n+2-id-j}\mathbf{V})\mathrm{Re}\,(a^{n-id-j+2}\mathbf{U}) \tag{55}$$
$$+P_{2j}(a^{n-id-j+2}\mathbf{U} + a^{m+n+2-id-j}\mathbf{V})\mathrm{Im}\,(a^{n-id-j+2}\mathbf{U})$$
$$+Q_{ij}(a^{n-id-j+2}\mathbf{U} + a^{n+m+2-id-j}\mathbf{V}) \quad (0 \le i \le p+q-1;\ 1 \le j \le d),$$

$$\sum_{i=0}^{p+q-1} Q_{ij}(\mathbf{U}) = K_j(\mathbf{U}) - p[P_{1j}(\mathbf{U})\mathrm{Re}\,(\mathbf{U}) + P_{2j}(\mathbf{U})\mathrm{Im}\,(\mathbf{U})] \quad (1 \le j \le d), \tag{56}$$

$$\sum_{j=1}^{d} K_j(\mathbf{U}) = \mathbf{O}, \tag{57}$$

where

$$P_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}') \quad (i = 1, 2; \quad 1 \le j \le d),$$
$$Q_{ij} : \mathcal{V} \to \mathcal{V}' \quad (0 \le i \le p+q-2; \quad 1 \le j \le d),$$
$$K_j : \mathcal{V} \to \mathcal{V}' \quad (1 \le j \le d-1)$$

are continuous functions.

We take into account (56) and (57) and we add together all equations (55). In this way we obtain (55) with

$$P_{1j}(\mathbf{U}) = (m+n)F_{1j}(\mathbf{U}), \quad P_{2j}(\mathbf{U}) = (m+n)F_{2j}(\mathbf{U}),$$
$$Q_{ij}(\mathbf{U}) = (m+n)G_{ij}(\mathbf{U}), \quad K_j(\mathbf{U}) = (m+n)H_j(\mathbf{U})$$
$$(0 \le i \le p+q-2; \quad 1 \le j \le d). \ \Box$$

Example 9. If $a^6 = 1$, then the general continuous solution of the functional equation

$$f(a^3\mathbf{Z}_1 + a^2\mathbf{Z}_2 + a\mathbf{Z}_3 + \mathbf{Z}_4,\ a\mathbf{Z}_5 + \mathbf{Z}_6) + f(a^3\mathbf{Z}_2 + a^2\mathbf{Z}_3 + a\mathbf{Z}_4 + \mathbf{Z}_5,\ a\mathbf{Z}_6 + \mathbf{Z}_1)$$
$$+f(a^3\mathbf{Z}_3 + a^2\mathbf{Z}_4 + a\mathbf{Z}_5 + \mathbf{Z}_6,\ a\mathbf{Z}_1 + \mathbf{Z}_2) + f(a^3\mathbf{Z}_4 + a^2\mathbf{Z}_5 + a\mathbf{Z}_6 + \mathbf{Z}_1,\ a\mathbf{Z}_2 + \mathbf{Z}_3)$$
$$+f(a^3\mathbf{Z}_5 + a^2\mathbf{Z}_6 + a\mathbf{Z}_1 + \mathbf{Z}_2,\ a\mathbf{Z}_3 + \mathbf{Z}_4) + f(a^3\mathbf{Z}_6 + a^2\mathbf{Z}_1 + a\mathbf{Z}_2 + \mathbf{Z}_3,\ a\mathbf{Z}_4 + \mathbf{Z}_5) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F_{11}(a\mathbf{U} + a^5\mathbf{V})\mathrm{Re}\,(a\mathbf{U}) + F_{21}(a\mathbf{U} + a^5\mathbf{V})\mathrm{Im}\,(a\mathbf{U})$$
$$+F_{11}(a^3\mathbf{U} + a\mathbf{V})\mathrm{Re}\,(a^3\mathbf{U}) + F_{21}(a^3\mathbf{U} + a\mathbf{V})\mathrm{Im}\,(a^3\mathbf{U})$$
$$-F_{11}(a^5\mathbf{U} + a^3\mathbf{V})\mathrm{Re}\,(a^5\mathbf{U} + 2a^3\mathbf{V}) - F_{21}(a^5\mathbf{U} + a^3\mathbf{V})\mathrm{Im}\,(a^5\mathbf{U} + 2a^3\mathbf{V})$$
$$+F_{12}(\mathbf{U} + a^4\mathbf{V})\mathrm{Re}\,(\mathbf{U}) + F_{22}(\mathbf{U} + a^4\mathbf{V})\mathrm{Im}\,(\mathbf{U})$$
$$+F_{12}(a^2\mathbf{U} + \mathbf{V})\mathrm{Re}\,(a^2\mathbf{U}) + F_{22}(a^2\mathbf{U} + \mathbf{V})\mathrm{Im}\,(a^2\mathbf{U})$$
$$-F_{12}(a^4\mathbf{U} + a^2\mathbf{V})\mathrm{Re}\,(a^4\mathbf{U} + 2a^2\mathbf{V}) - F_{22}(a^4\mathbf{U} + a^2\mathbf{V})\mathrm{Im}\,(a^4\mathbf{U} + 2a^2\mathbf{V})$$
$$+G_{01}(a\mathbf{U} + a^5\mathbf{V}) - G_{01}(a^5\mathbf{U} + a^3\mathbf{V}) + G_{02}(a\mathbf{U} + a^4\mathbf{V}) - G_{02}(a^4\mathbf{U} + a^2\mathbf{V})$$
$$+G_{11}(a^3\mathbf{U} + a\mathbf{V}) - G_{11}(a^5\mathbf{U} + a^3\mathbf{V}) + G_{12}(a^2\mathbf{U} + \mathbf{V}) - G_{12}(a^4\mathbf{U} + a^2\mathbf{V})$$
$$+H_1(a^5\mathbf{U} + a^3\mathbf{V}) - H_1(a^4\mathbf{U} + a^2\mathbf{V}),$$

where $F_{ij} : \mathcal{V} \to \mathcal{L}(\mathcal{V}^0, \mathcal{V}')$ $(i, j = 1, 2)$; $G_{ij} : \mathcal{V} \to \mathcal{V}'$ $(i = 0, 1;\ j = 1, 2)$ and $H_1 : \mathcal{V} \to \mathcal{V}'$ are arbitrary continuous complex vector functions.

Theorem 12. *If $a^{m+n}=1$ and $m=n$, the most general solution of the functional equation (40) is given by*

$$\begin{aligned} f(\mathbf{U},\mathbf{V}) &= \sum_{i=1}^{m}[F_1(a^i\mathbf{U},\ a^{n+i}\mathbf{V}) - F_i(a^i\mathbf{U},\ a^{n+i}\mathbf{V}) + H_i(a^{n+i}\mathbf{V}+a^i\mathbf{U})], \\ & \sum_{i=1}^{m} H_i(\mathbf{U}) = \mathbf{O}, \end{aligned} \tag{58}$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ $(1 \le i \le m)$ and $H_i : \mathcal{V} \to \mathcal{V}'$ $(1 \le i \le m-1)$ are arbitrary complex vector functions.

Proof. We start again from the equation (50). According to Theorem 3 and (49) we have

$$f(\mathbf{U},\mathbf{V}) = P_i(a^{m-i+2}\mathbf{U},\ a^{m+n+2-i}\mathbf{V}) \quad (1 \le i \le m),$$

$$f(\mathbf{U},\mathbf{V}) = Q_i(a^{m-i+2}\mathbf{U} + a^{m+n+2-i}\mathbf{V}) - P_i(a^{m+n+2-i}\mathbf{V},\ a^{m+2-i}\mathbf{U}) \quad (1 \le i \le m),$$

$$\sum_{i=1}^{m} Q_i(\mathbf{U}) = \mathbf{O}. \tag{59}$$

By addition we get (58) with

$$P_i(\mathbf{U},\mathbf{V}) = 2mF_{m-i+2}(\mathbf{U},\mathbf{V}), \quad Q_i(\mathbf{U}) = 2mH_{m-i+2}(\mathbf{U}). \ \square$$

Example 10. If $a^4 = 1$, the most general solution of the functional equation

$$f(a\mathbf{Z}_1+\mathbf{Z}_2,\ a\mathbf{Z}_3+\mathbf{Z}_4) + f(a\mathbf{Z}_2+\mathbf{Z}_3,\ a\mathbf{Z}_4+\mathbf{Z}_1)$$

$$+f(a\mathbf{Z}_3+\mathbf{Z}_4,\ a\mathbf{Z}_1+\mathbf{Z}_2) + f(a\mathbf{Z}_4+\mathbf{Z}_1,\ a\mathbf{Z}_2+\mathbf{Z}_3) = \mathbf{O}$$

is given by

$$f(\mathbf{U},\mathbf{V}) = F_1(a\mathbf{U},\ a^3\mathbf{V}) - F_1(a\mathbf{V},\ a^3\mathbf{U}) + F_2(a^2\mathbf{U},\ \mathbf{V})$$

$$-F_2(a^2\mathbf{V},\ \mathbf{U}) + H_1(a^3\mathbf{U}+a\mathbf{V}) - H_1(\mathbf{U}+a^2\mathbf{V}),$$

where $F_i : \mathcal{V}^2 \to \mathcal{V}'$ $(i = 1, 2)$ and $H_1 : \mathcal{V} \to \mathcal{V}'$ are arbitrary complex vector functions.

Now, as special cases we obtain the results given in [3,4,5].

References

[1] S. B. Prešić, D. Ž. Djoković, *Sur Une Equation Fonctionnelle*, Bull. Soc. Math. Phys. R. P. Serbie, **13**(1961), 149-152.

[2] D. Ž. Djoković, *A Special Cyclic Functional Equation*, Univ. Beograd. Publ. Elektroteh. Fak. Ser. Mat. Fiz., **143-155**(1965), 45-50.

[3] D. Ž. Djoković, R. Ž. Djordjević, P. M. Vasić, *On a Class of Functional Equations*, Publ. Inst. Math. Beograd, **6**(20)(1966), 65-76.

[4] R. Ž. Djordjević, P. M. Vasić, *O Jednoj Klasi Funkcionalnih Jednačina*, Mat. Vesnik, **4**(19)(1967), 33–38.

[5] D. S. Mitrinović, J. E. Pečarić, *Ciklične Nejednakosti i Ciklične Funkcionalne Jednačine*, Naučna Knjiga, Beograd 1991.

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