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A REPRESENTATION OF p-CONVEX SET-VALUED MAPS WITH VALUES IN $\mathbb{R}$

DORIAN POPA

Abstract. For a p-convex set-valued map with compact values in $\mathbb{R}$ is given a representation theorem as a sum of an additive function and a compact interval.

1. Introduction

Let X be a real vector space. We denote by $\mathcal{P}_0(X)$ the set of all nonempty subsets of X. A subset D of X is said to be *p-convex*, where p is a real number in the interval $(0, 1)$, if for every $x, y \in D$ we have:

$$(1-p)x + py \in D.$$

It is known (see [4]) that every p-convex and closed subset of a real topological vector space is a convex set. A $\frac{1}{2}$-convex set is called *midconvex* set.

Let D be a p-convex and nonempty subset of X. A set-valued map $F : D \to \mathcal{P}_0(\mathbb{R})$ is said to be *p-convex* if for every $x, y \in D$ we have:

$$(1-p)F(x) + pF(y) \subseteq F((1-p)x + py).$$

A function $f : D \to \mathbb{R}$ is said to be *p-convex* (*concave*) if for every $x, y \in D$ we have:

$$f((1-p)x + py) \leq (\geq)(1-p)f(x) + pf(y).$$

The following assertions, which are true for midconvex set-valued maps and functions [3], holds for p-convex set-valued maps and functions.

A set valued map $F : D \to \mathcal{P}_0(\mathbb{R})$ is p-convex if and only if the *graph* of F, defined by

$$Graph\, F = \{(x, y) \in X \times \mathbb{R} : \ y \in F(x)\},$$

is a p-convex subset of the vector space $X \times \mathbb{R}$.

A function $f : D \to \mathbb{R}$ is p-convex if and only if the *epigraph* of f, defined by

$$Epi\, f = \{(x, t) \in X \times \mathbb{R} : \ f(x) \leq t\},$$

is a p-convex subset of the vector space $X \times \mathbb{R}$.

Example 1.1. Let $f, g : D \to \mathbb{R}$ be two functions such that $f(x) \leq g(x)$ for every $x \in D$. Then the set valued map $F : D \to \mathcal{P}_0(\mathbb{R})$ given by the relation

$$F(x) = [f(x), g(x)]$$

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for every $x \in D$ is p-convex if and only if f is p-convex and g is p-concave.
Proof. Let $x, y \in D$. We have

$$(1-p)F(x) + pF(y) = [(1-p)f(x) + pf(y), (1-p)g(x) + pg(y)]$$

and

$$F((1-p)x + py) = [f((1-p)x + py), g((1-p)x + py)].$$

The relation

$$(1-p)F(x) + pF(y) \subseteq F((1-p)x + py)$$

holds if and only if we have

$$f((1-p)x + py) \leq (1-p)f(x) + pf(y)$$

and

$$(1-p)g(x) + pg(y) \leq g((1-p)x + py),$$

hence f is p-convex and g is p-concave.
Remark 1.1. If $F : D \to \mathcal{P}_0(\mathbb{R})$ is a p-convex set-valued map with closed values, then it is convex valued.
Proof. Let $x \in D$. We have

$$(1-p)F(x) + pF(x) \subseteq F((1-p)x + px) = F(x),$$

hence $F(x)$ is a p-convex subset of $\mathbb{R}$ and being closed it is a convex subset of $\mathbb{R}$. □

The goal of this paper is to give a representation of p-convex set-valued maps with compact values in $\mathbb{R}$. For additive set-valued function this problem was studied by H. Rädstrom [8]. Later K. Nikodem [5], gave a characterization of midconvex set-valued maps with compact values in $\mathbb{R}$. A representation of the solutions of a generalization of Jensen equation for set-valued maps is given by the author in [7]. K. Nikodem, F. Papalini and S. Vercillo [6], established conditions under which every midconvex set-valued function can be represented as a sum of an additive function and a convex set-valued function. We prove that an analogous result holds for p-convex set-valued maps with compact values in $\mathbb{R}$.

2. Main results

For the characterization of p-convex set-valued maps with compact values in $\mathcal{P}_0(\mathbb{R})$ we need some lemmas.
Lemma 2.1. ([2]) *Let $p \in (0,1)$. Denote by $(P_n)_{n\geq 1}$ the sequence of sets defined as follows: $P_1 = \{0, p, 1\}$; if $P_n = \{0, p_n^{(1)}, \ldots, p_n^{(2^n-1)}\}$, where*

$$0 < p_n^{(1)} < \cdots < p_n^{(2^n-1)} < 1,$$

is defined, put

$$P_{n+1} = P_n \cup \{(1-p)p_n^{(k-1)} + pp_n^{(k)} : \ 1 \leq k \leq 2^n\}$$

where $p_0^{(0)} = 0$ and $p_n^{(2n)} = 1$. Then the set

$$P = \bigcup_{n\geq 1} P_n$$

is dense in the interval $[0,1]$.

Lemma 2.2. ([2]) *Let X be a real linear space and D a p-convex and nonempty subset of X. Then D is q-convex for each $q \in P$, where P is the set defined in Lemma 2.1.*

Lemma 2.3. *Let X be a real linear space, D a p-convex and nonempty subset of X. If a set-valued map $F : D \to \mathcal{P}_0(Y)$ is p-convex then it is q-convex for every $q \in P$, where P is the set defined in Lemma 2.1.*

Proof. From the p-convexity of F it results that $Graph\, F$ is a p-convex subset of $X \times \mathbb{R}$, and using Lemma 2.2 we obtain that $Graph\, F$ is q-convex for every $q \in P$. Then F is q-convex for every $q \in P$. □

Theorem 2.1. *Let D be a linear subspace of the real linear space X and $F : D \to \mathcal{P}_0(\mathbb{R})$ be a p-convex set-valued map with bounded values. Then there exists an additive function $a : D \to \mathbb{R}$ and two real numbers s, t, $s \leq t$, such that for every $x \in D$*

$$a(x) + s \leq F(x) \leq a(x) + t.$$

Proof. Following the method used in [5], for any $x \in D$ put $f(x) = \inf F(x)$ and $g(x) = \sup F(x)$. Then $f : D \to \mathbb{R}$ is p-convex and $g : D \to \mathbb{R}$ is p-concave. Indeed, for every $x, y \in X$ we have:

$$\begin{aligned} f((1-p)x + py) &= \inf F((1-p)x + py) \\ &\leq \inf((1-p)F(x) + pF(y)) \\ &= \inf((1-p)F(x) + \inf(pF(y)) \\ &= (1-p)f(x) + pf(y), \end{aligned}$$

hence f is a p-convex function and analogously g is a p-concave function. We have also

$$f(x) \leq F(x) \leq g(x)$$

for every $x \in D$.

Let $h : D \to \mathbb{R}$, $h(x) = g(x) - f(x)$ for every $x \in D$. Obviously h is p-concave and $h(x) \geq 0$ for every $x \in D$. We prove that h is a constant function.

The function $-h$ is p-convex, hence the set $Epi(-h)$ is p-convex and it follows from Lemma 2.2 that $Epi(-h)$ is q-convex for every $q \in P$. It follows that $-h$ is a q-convex function for $q \in P$, hence h is q-concave for $q \in P$.

Suppose that h is nonconstant. Then there exist $x, y \in X$, $x \neq y$, such that $h(x) < h(y)$. Using the density of P in $[0, 1]$ it follows that there exists $t > 1$, $\frac{1}{t} \in P$, such that:

$$t(h(x) - h(y)) + h(y) < 0.$$

From the q-concavity of h with $q \in P$ we get:

$$\begin{aligned} h(x) &= h\left(\frac{1}{t}(tx + (1-t)y) + \left(1 - \frac{1}{t}\right) y\right) \\ &\geq \frac{1}{t} h(tx + (1-t)y) + \left(1 - \frac{1}{t}\right) h(y) \end{aligned}$$

and foreward it follows

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y) = t(h(x) - h(y)) + h(y) < 0,$$

contradiction with nonnegativity of the values of h.

Hence there exists $c \in \mathbb{R}$ such that $h(x) = c$, for every $x \in X$.

The function $f = g - c$ is p-concave and being p-convex satisfies the relation

$$f((1-p)x + py) = (1-p)f(x) + pf(y). \tag{1}$$

We prove that there exists an additive function $a : D \to \mathbb{R}$ and $k \in \mathbb{R}$ such that $f(x) = a(x) + k$ for every $x \in D$.

For $x = 0$ and $y \in D$ in (1) we have

$$f(py) = pf(y) + (1-p)f(0). \tag{2}$$

For $y = 0$ and $x \in D$ in (1) we have

$$f((1-p)x) = (1-p)f(x) + pf(0). \tag{3}$$

Let $u, v \in D$. From (1), (2), (3) we have

$$\begin{aligned} f(u+v) &= f\left((1-p)\frac{u}{1-p} + p\frac{v}{p}\right) \\ &= (1-p)f\left(\frac{u}{1-p}\right) + pf\left(\frac{v}{p}\right) \\ &= (1-p)f\left(\frac{u}{1-p}\right) + pf(0) + pf\left(\frac{v}{p}\right) \\ &+ (1-p)f(0) - (1-p)f(0) - pf(0) \\ &= f(u) + f(v) - f(0). \end{aligned}$$

The function $a : D \to \mathbb{R}$, $a(x) = f(x) - f(0)$, $x \in D$, is additive. Indeed for any $x, y \in X$ we have:

$$a(x+y) = f(x+y) - f(0) = f(x) + f(y) - f(0) - f(0) = a(x) + a(y).$$

Denoting $s = f(0)$ we obtain $f(x) = a(x) + s$ and $g(x) = a(x) + t$ for every $x \in D$, where $t = s + c$. □

Corollary 2.1. *Let D be a linear subspace of a real linear space X and $F : D \to \mathcal{P}_0(\mathbb{R})$ be a p-convex set-valued map with compact values.*

Then there exists an additive function $a : D \to \mathbb{R}$ and a compact interval I in $\mathbb{R}$ such that

$$F(x) = a(x) + I$$

for every $x \in D$.

Proof. In view of Theorem 2.1, there exist an additive function $a : D \to \mathbb{R}$ and $s, t \in \mathbb{R}$, $s \le t$, such that

$$a(x) + s \le F(x) \le a(x) + t$$

for every $x \in D$. Taking account of the Remark 1.1, $F(x)$ is a convex subset of $\mathbb{R}$, hence

$$F(x) = [a(x) + s, a(x) + t] = a(x) + I$$

for every $x \in D$, where $I = [s, t]$. □

Remark 2.1. If p is a rational number in the interval $(0, 1)$ then the converse of Corollary 2.1 is true.

Proof. Let $a : D \to \mathbb{R}$ be an additive function, I a compact interval in $\mathbb{R}$ and $F(x) = a(x)+I$ for every $x \in D$. Taking into account that a is rationally homogeneous [1] it follows that

$$\begin{aligned} F((1-p)x+py) &= a((1-p)x+py) \\ &= (1-p)a(x)+pa(y)+(1-p)I+pI \\ &= (1-p)F(x)+pF(y) \end{aligned}$$

for every $x, y \in D$. □

The results proved in Theorem 2.1 and Corollary 2.1 are extensions of the results obtained in [5] for midconvex-valued maps.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Academic Press, New York and London, 1966.
[2] A. Aleman, *On some generalizations of convex sets and convex functions*, Mathematica. Rev. d'anal. numér. théor. de l'approx., **1**(1985), 1-6.
[3] J. P. Aubin, H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston-Basel-Berlin, 1990.
[4] W. E. Breckner, G. Kassay, *A sistematization of convexity concepts for sets and functions*, Journal of convex analysis, **1**(1997), 109-127.
[5] K. Nikodem, *Midpoint convex functions majorized by midpoint concave functions*, Aequationes Math., **32**(1987), 45-51.
[6] K. Nikodem, F. Papalini, S. Vercillo, *Some representations of midconvex set-valued functions*, Aequationes Math., **53**(1997), 127-140.
[7] D. Popa, *Set-valued solutions for an equation of Jensen type*, Rev. d'anal. numér. théor. de l'approx., Tome 28, No. **1**(1999), 73-77.
[8] H. Radström, *One-parameter semigroups of subsets of a real linear space*, Ark. Mat., **4**(1960), 87-97.

Department of Mathematics, Str. C. Daicoviciu, 15,
3400 Cluj-Napoca, Romania
E-mail address: Popa.Dorian@math.utcluj.ro