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SOME SUFFICIENT CONDITIONS FOR UNIVALENCE

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Abstract. In this paper we prove the analyticity and the univalence of the functions which are defined by means of integral operators. In particular cases we find some known results.

1. Introduction

We denote by $U_r = \{z \in C : |z| < r\}$ the disk of z-plane, where $r \in (0, 1]$, $U_1 = U$ and $I = [0, \infty)$.

Let A be the class of functions f analytic in U such that $f(0) = 0,\ f'(0) = 1$. Let S denote the class of function $f \in A,\ f$ univalent in U. The usual subclasses of S consisting of starlike functions and α-convex functions will be denoted by S^* respectively M_α.

Definition 1.1. *([2])Let $f \in A,\ f(z)f'(z) \neq 0$ for $0 < |z| < 1$ and let $\alpha \geq 0$. We denote by*

$$M(\alpha, f) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right) \tag{1}$$

If $ReM(\alpha, f) > 0$ in U, then f is said to be an α- convex function ($f \in M_\alpha$).

Theorem 1.1. *([2]). The function $f \in M_\alpha$ if and only if there exists a function $g \in S^*$ such that*

$$f(z) = \left(\frac{1}{\alpha}\int_0^z \frac{g^{\frac{1}{\alpha}}(u)}{u}du\right)^\alpha \tag{2}$$

Definition 1.2. *([5]) Let $f \in A$. We said that $f \in S^*(a, b)$ if*

$$\left|\frac{zf'(z)}{f(z)} - a\right| < b, \qquad |z| < 1, \tag{3}$$

where

$$a \in C,\ Rea \geq b,\ |a - 1| < b. \tag{4}$$

Theorem 1.2. *([1]) Let $f \in A$. If for all $z \in U$*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{5}$$

then the function f is univalent in U.

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2. Preliminaries

Theorem 2.1. *([4]) Let $L(z,t) = a_1(t)z + a_2(t)z^2 + \dots,\ a_1(t) \neq 0$ be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \quad \forall z \in U_r, \tag{6}$$

where $p(z,t)$ is analytic in U and satisfies the condition $Rep(z,t) > 0$ for all $z \in U$, $t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then for each $t \in I$ the function $L(z,t)$ has an analytic and univalent extension to the whole disk U.

3. Main results

Theorem 3.1. *Let $f,\ g \in A$ and let α be a complex number, $|\alpha - 1| < 1$. If*

$$(1-|z|^2)\left|(\alpha-1)\frac{zg'(z)}{g(z)} + \frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \forall z \in U, \tag{7}$$

then the function

$$H(z) = \left(\alpha\int_0^z g^{\alpha-1}(u)f'(u)du\right)^{1/\alpha} \tag{8}$$

is analytic and univalent in U.

Proof. Let us prove that there exists a real number $r \in (0,1]$ such that the function $L : U_r \times \to C$ defined formally by

$$L(z,t) = \left[\int_0^{e^{-t}z} g^{\alpha-1}(u)f'(u)du + (e^t - e^{-t})zg^{\alpha-1}(e^{-t}z)f'(e^{-t}z)\right]^{1/\alpha} \tag{9}$$

is analytic in U_r for all $t \in I$.
Since $g \in A$, the function $h(z) = \frac{g(z)}{z}$ is analytic in U and $h(0) = 1$. Then there is a disk U_{r_1}, $0 < r_1 \leq 1$, in which $h(z) \neq 0$ for any $z \in U_{r_1}$ and we choose the uniform branch of $(h(z))^{\alpha-1}$ equal to 1 at the origin, denoted by h_1.
For the function

$$h_2(t) = \int_0^{e^{-t}z} u^{\alpha-1}h_1(u)f'(u)du$$

we have $h_2(z,t) = z^\alpha h_3(z,t)$ and is easy to see that h_3 is also analytic in U_{r_1}. The function

$$h_4(z,t) = h_3(z,t) + (e^t - e^{-t})e^{-t(\alpha-1)}h_1(e^{-t}z)f'(e^{-t}z)$$

is analytic in U_{r_1} and we have $h_4(0,t) = e^{(2-\alpha)t}[1 + (1/\alpha - 1)e^{-2t}] \neq 0$ for any $t \in I$. It results that there exist $r_2 \in (0, r_1]$ such that $h_4(z,t) \neq 0$ in U_{r_2}. Then we can choose an uniform branch of $[h_4(z,t)]^{1/\alpha}$ analytic in U_{r_2} denoted by $h_5(z,t)$, which is equal to $a_1(t) = e^{(2-\alpha)t/\alpha}[1 + (1/\alpha - 1)e^{-2t}]^{1/\alpha}$ at the origin and for $a_1(t)$ we fix a determination.
From this considerations it results that the relation (9) may be written as

$$L(z,t) = zh_5(z,t) = a_1(t)z + a_2(t)z^2 + \dots$$

and then the function $L(z,t)$ is analytic in U_{r_2}.
Since $|\alpha - 1| < 1$ it results that $\lim_{n\to\infty}|a_1(t)| = \infty$.

It is easy to prove that $L(z,t)$ is locally absolutely continous in I, locally uniformly with respect to U_{r_3} and that $\{L(z,t)/a_1(t)\}$ is a normal family in $U_{r_3},\ r_3 \in (0, r_2]$. It follows that the function $p(z,t)$ defined by (6) is analytic in $U_r,\ r \in (0, r_3]$, for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \in I$, it is sufficient to prove that the function $w(z,t)$ defined in U_r by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1}$$

can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

After computation we obtain

$$w(z,t) = (1 - e^{-2t}) \left[(\alpha - 1) \frac{e^{-t} z g'(e^{-t}z)}{g(e^{-t}z)} + \frac{e^{-t} z f''(e^{-t}z)}{f'(e^{-t}z)} \right] \tag{10}$$

From (7) we deduce that the function $w(z,t)$ is analytic in the unit disk U. We have $w(z,0) = 0$ and also $|w(0,t)| = |(1 - e^{-2t})(\alpha - 1)| < |\alpha - 1| < 1$.

Let us denote $u = e^{-t} e^{i\theta}$. Then $|u| = e^{-t}$ and taking into account the relation (7) we have

$$|w(e^{i\theta}, t)| = (1 - |u|^2) \left| (\alpha - 1) \frac{u g'(u)}{g(u)} + \frac{u f''(u)}{f'(u)} \right| \leq 1$$

Using the maximum principle for all $z \in U \setminus \{0\}$ and $t > 0$ we conclude that $|w(z,t)| < 1$ and finally we have $|w(z,t)| < 1$ for all $z \in U$ and $t \in I$.

From Theorem 2.1 it results that the function

$$L(z,0) = \left(\int_0^z g^{\alpha-1}(u) f'(u) du \right)^{1/\alpha}$$

is analytic and univalent in U and then the function H defined by (8) is analytic and univalent in U.

For particular choices of f and g we get the following

Corollary 3.1. *Let $f \in A$ and let $\alpha \in C,\ |\alpha - 1| < 1$. If*

$$(1 - |z|^2) \left| \frac{z f''(z)}{f'(z)} \right| \leq 1 - |\alpha - 1|(1 - |z|^2), \quad \forall z \in U, \tag{11}$$

then the function

$$F(z) = \left(\alpha \int_0^z u^{\alpha-1} f'(u) du \right)^{1/\alpha} \tag{12}$$

is analytic and univalent in U.

Proof. For the function $g(z) = z$, from (7) we have

$$(1 - |z|^2) \left| \frac{z f''(z)}{f'(z)} + \alpha - 1 \right| \leq 1$$

We observe that if the condition (5) of Theorem 1.2 will be replaced by the strong condition (11), then we have not only the univalence of the function f $(\alpha = 1)$, but we obtain also the univalence of the function F defined by (12).

Corollary 3.2. *Let $g \in A$ and let $\alpha \in C,\ |\alpha - 1| < 1$. If*

$$(1-|z|^2)\left|(\alpha-1)\frac{zg'(z)}{g(z)}\right| \le 1, \quad \forall z \in U, \tag{13}$$

then the function

$$G(z) = \left(\alpha \int_0^z g^{\alpha-1}(u)du\right)^{1/\alpha} \tag{14}$$

is analytic and univalent in U.

Proof. If we take $f(z) = z$, from (7) we obtain the relation (13). So we find a result from paper [3].

For the function $f \in A,\ f'(z) = \frac{g(z)}{z}$, from theorem 3.1 we get the following

Theorem 3.2. *Let $g \in A$ and let $\alpha \in C,\ |\alpha - 1| < 1$. If*

$$(1-|z|^2)\left|\alpha\frac{zg'(z)}{g(z)} - 1\right| \le 1, \quad \forall z \in U, \tag{15}$$

then the function

$$G(z) = \left(\alpha \int_0^z \frac{g^{\alpha}(u)}{u}du\right)^{1/\alpha} \tag{16}$$

is analytic and univalent in U.

The operator (16) is just the integral operator introduced by Prof. P. T. Mocanu in the integral representation of α-convex functions.

Corollary 3.3. *Let $g \in A,\ \alpha \in C,\ |\alpha - 1| < 1$. If*

$$\left|\frac{zg'(z)}{g(z)} - \frac{1}{\alpha}\right| \le \frac{1}{|\alpha|}, \qquad \forall z \in U,$$

then the function G defined by (16) is analytic and univalent in U.

Remark . Let $\alpha \in (0,\ 2)$ and let $g \in S^*(\frac{1}{\alpha}, \frac{1}{\alpha})$. Then the function G defined by (16) is analytic and univalent in U.

Indeed, if we consider $a = 1/\alpha$ and $b = 1/|\alpha|$, the conditions (4) are satisfied for $\alpha \in (0,\ 2)$. But here we obtain only the univalence of G.

If in theorem 3.1 we take $f \equiv g$,we have

Theorem 3.3. *Let $f \in A$ and let $\alpha \in C,\ |\alpha - 1| < 1$. If*

$$(1-|z|^2)\left|(\alpha-1)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)}\right| \le 1, \qquad \forall z \in U \tag{17}$$

then the function f is univalent in U.

Corollary 3.4. *Let $f \in A,\ \beta \in C,\ Re\beta > \frac{1}{2}$. If*

$$|\ M(\ \beta,\ f\ ) - \beta\ | \le |\beta| \tag{18}$$

for all $z \in U$, then the function f is univalent in U.

Proof. For $\beta = 1/\alpha$, from $|\alpha - 1| < 1$ we get $\mathrm{Re}\beta > \frac{1}{2}$ and

$$\left(\frac{1}{\beta} - 1\right)\frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} = \frac{1}{\beta}\,[\ M(\beta,\ f) - \beta\ ]$$

Remark. In the case $\beta > \frac{1}{2}$, the condition (18) implies $\mathrm{Re}M(\beta,\ f) > 0$ and from Theorem 1.1 we get that f is a β-convex function.

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