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ON A PARTICULAR FIRST ORDER NONLINEAR DIFFERENTIAL SUBORDINATION II

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Abstract. We find conditions on the complex-valued functions B, C, D in unit disc U and the positive constants M and N such that

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies $|p(z)| < N$, where p is analytic in U, with $p(0) = 0$.

1. Introduction and preliminaries

We let $\mathcal{H}[U]$ denote the class of holomorphic functions in the unit disc

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}[U], \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\}$$

and

$$\mathcal{A}_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots, \ z \in U\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

We let Q denote the class of functions q that are holomorphic and injective in $\overline{U} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial U : \ \lim_{z \to \zeta} q(z) = \infty\right\}$$

and furthermore $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$, where $E(q)$ is called exception set.

In order to prove the new results we shall use the following:

Lemma A. [1] (Lemma 2.2.d p. 24) *Let $q \in Q$, with $q(0) = a$, and let*

$$p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$, $r_0 < 1$ and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

(i) $p(z_0) = q(\zeta_0)$

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, *and*

(iii) $\mathrm{Re}\dfrac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \mathrm{Re}\left[\dfrac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right].$

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In [1] chapter IV, the authors have analyzed a first-order linear differential subordination

$$B(z)zp'(z) + C(z)p(z) + D(z) \prec h(z), \tag{1}$$

where B, C, D and h are complex-valued functions in the unit disc U. A more general version of (1) is given by

$$B(z)zp'(z) + C(z)p(z) + D(z) \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{C}$.

In [2] we found conditions on the complex-valued functions B, C, D, E in the unit disc U and the positive constants M and N such that

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z) + E(z)| < M$$

implies $|p(z)| < N$, where $p \in \mathcal{H}[0, n]$.

In this paper we shall consider a particular first-order nonlinear differential subordination given by the inequality

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M \tag{3}$$

We find conditions on the complex-valued functions B, C, D such that (3) implies $|p(z)| < N$ where $p \in \mathcal{H}[0, n]$.

In some cases, given the functions B, C, D and the constant M we will determine an appropriate N such that (3) implies $|p(z)| < N$.

2. Main results

The results in [2] can certainly be used in the special case when $E(z) \equiv 0$. However, in this case we can improve those results by the following theorem:

Theorem. *Let $M > 0$, $N > 0$, and let n be a positive integer. Suppose that the functions $B, C, D : U \to \mathbb{C}$ satisfy $B(z) \neq 0$,*

$$\begin{cases} (i)\ \mathrm{Re}\dfrac{D(z)}{B(z)} \geq -n \\ (ii)\ |nB(z) + D(z)| \geq \dfrac{1}{N}[M + N^2|C(z)|]. \end{cases} \tag{4}$$

If $p \in \mathcal{H}[0, n]$ and

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M \tag{5}$$

then

$$|p(z)| < N.$$

Proof. If we let

$$W(z) = B(z)zp'(z) + C(z)p^2(z) + D(z)p(z), \tag{6}$$

then from (6) we obtain

$$|W(z)| = |B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)|. \tag{7}$$

From (7) and (5) we have

$$|W(z)| < M, \quad z \in U. \tag{8}$$

Assume that $|p(z)| \not< N$, which is equivalent with $p(z) \not\prec Nz = q(z)$.

According to Lemma A, with $q(z) = Nz$, there exist $z_0 \in U$, $z_0 = r_0 e^{i\theta_0}$, $r_0 < 1$, $\theta_0 \in [0, 2\pi)$, $\zeta \in \partial U$, $|\zeta| = 1$ and $m \geq n$, such that $p(z_0) = N\zeta$ and $z_0 p'(z_0) = mN\zeta$.

Using these conditions in (7) we obtain for $z = z_0$

$$|W(z_0| = |B(z_0)mN\zeta + C(z_0)N^2\zeta + D(z_0)N\zeta| = \tag{9}$$
$$= |N[B(z_0)m + D(z_0)] + C(z_0)N^2\zeta| \geq$$
$$\geq N|B(z_0)m + D(z_0)| - N^2|C(z_0)|.$$

Since $m \geq n$ and $B(z) \neq 0$, from condition (i) we have

$$\left| m + \frac{D(z)}{B(z)} \right| \geq \left| n + \frac{D(z)}{B(z)} \right|,$$

and

$$|mB(z) + D(z)| \geq |nB(z) + D(z)|.$$

For $z = z_0$, we have

$$|mB(z_0) + D(z_0)| \geq |nB(z_0) + D(z_0)|.$$

Using this last result and condition (ii) together with (9) we deduce that

$$|W(z_0)| \geq N[nB(z_0) + D(z_0)| - N^2|C(z_0)| \geq M.$$

Since this contradicts (8) we obtain the desired result $|p(z)| < N$. □

Instead of prescribing the constant N in Theorem, in some cases we can use (ii) to determine an appropriate $N = N(M, n, B, C, D)$ so that (5) implies $|p(z)| < N$. This can be accomplished by solving (ii) for N and by taking the supremum of the resulting function over U. The conditions (ii) is equivalent to

$$|C(z)|N^2 - N|nB(z) + D(z)| + M \leq 0. \tag{10}$$

The inequality (10) holds if:

$$|nB(z) + D(z)|^2 \geq 4|C(z)|. \tag{11}$$

In this case we let

$$N = \sup_{|z|<1} \frac{|nB(z) + D(z)| - \sqrt{|nB(z) + D(z)|^2 - 4M|C(z)|}}{2|C(z)|} =$$
$$= \sup_{|z|<1} \frac{2M}{|nB(z) + D(z)| + \sqrt{|nB(z) + D(z)|^2 - 4M|C(z)|}}$$

If this supremum is finite, we have the following version of the Theorem:

Corollary. *Let $M > 0$ and let n be a positive integer. Suppose that $p \in \mathcal{H}[0, n]$, and the functions $B, C, D : U \to \mathbb{C}$, with $B(z) \neq 0$, $C(z) \neq 0$, satisfy:*

$$\text{Re}\left[\frac{D(z)}{B(z)}\right] \geq -n, \quad |nB(z) + D(z)| \geq 4|C(z)|$$

and let

$$N = \sup_{|z|<1} \frac{2M}{|nB(z) + D(z)| + \sqrt{|nB(z) + D(z)|^2 - 4M|C(z)|}} < \infty$$

Then

$$|B(z)zp'(z) + C(z)p^2(z) + D(z)p(z)| < M$$

implies

$$|p(z)| < N.$$

If $n = 1$, $B(z) = 3 + z$, $C(z) = 1$, $D(z) = 1 - z$, $M = 1$, $N = 2 - \sqrt{3}$.

In this case from Corollary, we deduce

Example 1. If $p \in \mathcal{H}[0, 1]$, then

$$|(3 + z)zp'(z) + p^2(z) + (1 - z)p(z)| < 1$$

implies

$$|p(z)| < 2 - \sqrt{3}.$$

If $n = 3$, $B(z) = 1 + z$, $C(z) = 2$, $D(z) = 4 - 3z$, $M = 2$, $N = \dfrac{7 - \sqrt{33}}{4}$. In this case from Corollary, we deduce:

Example 2. If $p \in \mathcal{H}[0, 3]$, then

$$|(1 + z)zp'(z) + 2zp^2(z) + (4 - 3z)p(z)| < 2$$

implies

$$|p(z)| < \frac{7 - \sqrt{33}}{4}.$$

References

[1] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[2] Gh. Oros and Georgia Irina Oros, *On a particular first-order nonlinear differential subordination I*, (to appear).

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