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ON LACUNARY INVARIANT SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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Abstract. The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of lacunary convergence, invariant mean and the sequence of modulus functions We also examine some topological properties of these spaces.

1. **Introduction**

Let ℓ_∞ and $c$denote the Banach spaces of real bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or σ-mean if and only if

i. $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
ii. $\phi(e) \geq 0$, where *e= (1,1,1,..)* and,
iii. $\phi\left(x_{\sigma(n)}\right) = \phi(x)$ for all $x \in \ell_\infty$.

Let V_σ denote the set of bounded sequence all of whose invariant means are equal. In particular, if σ is the translation $n \to n+1$, then a σ-mean reduce to a Banach limit (see, Banach [1]) and set V_σ reduce to $\hat{c}$, the spaces of all almost convergent sequences (see, Lorentz [7]).

If $x = (x_n)$, write $Tx = Tx_n = \left(x_{\sigma(n)}\right)$. It can be shown (Schaefer [16]) that $V_\sigma = \left\{ x \in \ell_\infty : \lim_k t_{kn}(x) = \ell, \text{ uniformly in n,} \right\}$ $\ell = \sigma - \lim x$, where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + ... + x_{\sigma^k(n)}}{k+1}.$$

Here $\sigma^k(n)$ denote the k^{th} iterate of the mapping σ at n. The mapping σ is one to one and such that $\sigma^k(n) \neq n$ for all positive integers n and k. Thus a σ- mean ϕ extends the limit functional on c, the spaces of convergent sequence, in the sense that $\phi(x) = \lim x$ for all $x \in c$. (see, Mursaleen [11]).

We call V_σ as the space of σ-convergent sequences.

A sequence $x = (x_k)$ is said to be strongly σ-convergent (Mursallen [12]) if there exists a number ℓ such that $\lim_k \frac{1}{k} \sum_{j=1}^{k} \left| x_{\sigma^j(n)} - \ell \right| = 0$ uniformly in n.

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We denote $[V_\sigma]$ as the set of all strongly σ-convergent sequences. In case $\sigma(n) = n+1$, $[V_\sigma]$ reduce to $[\hat{c}]$, the space of all strong almost convergent sequence (Maddox [8]).

Also the strongly almost convergent sequences was studied by Freedman et all [4], independently.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, \mathrm{k_r}]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al [4] as:

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - \ell| = 0, \text{for some}\, \ell \right\}$$

Recently, the concept of lacunary strong σ-convergence was introduced by Savas [14] which is a generalization of the idea of lacunary strong almost convergence due to Das and Mishra [2].

A modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i. $f(x) = 0$ if and only if $x = 0$
ii. $f(x+y) \leq f(x) + f(y)$, for all $x, y > 0$
iii. f is increasing,
iv. f is continuous from the right at zero.

Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from conditions (ii) and (iv) that f is continuous everywhere on $[0,\infty)$.

A modulus function may be bounded or unbounded. For example, $f(t) = \frac{t}{t+1}$ is bounded but $f(t) = t^p \; (0 < p \leq 1)$ is unbounded.

Ruckle [13] and Maddox [9], Savas [15] and other authors used modulus function to construct new sequence spaces.

Recently, Kolk ([6], [7]) gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ i.e.,

$$X(f_k) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$$

In this paper by combining lacunary sequence , invariant mean and a sequence of modulus functions, we define the following new sequence spaces:

$$\left[w_\sigma^0, F\right]_\theta = \left\{ x : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|t_{kn}(x)|) = 0, \text{uniformly in n} \right\}$$

$$[w_\sigma, F]_\theta = \left\{ x : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|t_{kn}(x - l)|) = 0, \text{uniformly in n, for some l} \right\}$$

$$[w_\sigma^\infty, F]_\theta = \left\{ x : \sup_{r,n} \frac{1}{h_r} \sum k \in I_r f_k(|t_{kn}(x)|) < \infty \right\}$$

$$[w_\sigma, F] = \left\{ x : \lim_r \frac{1}{m} \sum_{k=1}^{m} f_k(|t_{kn}(x - l)|) = 0, \text{uniformly in n, for some l} \right\}$$

Some sequence spaces are obtained by specializing F, θ, σ. For example, if

$\theta = (2^r)$, $\sigma(n) = n+1$and $f_k(x) = x$ for all k, then $[w_\sigma, F]_\theta = \hat{w}$ (see, Das and Sahoo [3]). If $\sigma(n) = n+1$ and $f_k(x) = f$ for all k, then $[w_\sigma, F]_\theta = [\hat{w}(f)]_\theta$ and $[w_\sigma, F] = [\hat{w}(f)]$ (see, Mursaleen and Chishti [12]).

When $\sigma(n) = n+1$, the spaces $\left[w_\sigma^0, F\right]_\theta$, $[w_\sigma, F]_\theta$ and $[w_\sigma^\infty, F]_\theta$ reduce to the spaces $[\hat{w}_0, F]_\theta$, $[\hat{w}, F]_\theta$ and $[\hat{w}_\infty, F]_\theta$ respectively, where

$$[\hat{w}, F]_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} f_k(|d_{kn}(x - l)|) = 0, \right.$$

$$\left. \text{uniformly in n, for some l} \right\}$$

and

$$d_{nk}(x) = \frac{x_n + x_{n+1} + ... + x_{n+k}}{k+1}$$

If $\theta = (2^r)$, then $\left[w_\sigma^0, F\right]_\theta = \left[w_\sigma^0, F\right]$, $[w_\sigma, F]_\theta = [w_\sigma, F]$ and $[w_\sigma^\infty, F]_\theta = [w_\sigma^\infty, F]$.

2. Main Results

We have

Theorem 2.1. *For any a sequence of modulus functions $F = (f_k)$, $\left[w_\sigma^0, F\right]_\theta$, $[w_\sigma, F]_\theta$, $[w_\sigma^\infty, F]_\theta$ and $[w_\sigma, F]$ are linear spaces over the set of complex numbers.*

Proof. We shall prove the result only for $\left[w_\sigma^0, F\right]_\theta$. The others can be treated similarly. Let $x, y \in \left[w_\sigma^0, F\right]_\theta$and $\alpha, \beta \in$ C. Then there exist integers H_α and $K_\beta$such that $|\alpha| < H_\alpha$ and $|\beta| < K_\beta$. We have

$$\frac{1}{h\,r} \sum_{k \in I_r} f_k(|t_{kn}(\alpha x - \beta y)|) \leq H_\alpha \frac{1}{h_r} \sum_{k \in I_r} f_k(|t_{kn}(x)|)$$

$$+ K_\beta \frac{1}{h_r} \sum_{k \in I_r} f_k(|t_{kn}(y)|)$$

This implies $\alpha x + \beta y \in \left[w_\sigma^0, F\right]_\theta$ □

We will now give a lemma.

Lemma 2.2. *Let f be a modulus and let $0 < \delta < 1$. Then for each $|t_{kn}(x)| > \delta$ for all k and n we have*

$$f(|t_{kn}(x)|) \leq 2f(1)\,\delta^{-1}\,|t_{kn}(x)|$$

Proof.

$$f(|t_{kn}(x)|) \leq f\left(1 + \left[\frac{|t_{kn}(x)|}{\delta}\right]\right) \leq f(1) + f\left(\left[\frac{|t_{kn}(x)|}{\delta}\right]\right)$$

$$\leq f(1)\left(1 + \frac{|t_{kn}(x)|}{\delta}\right) \leq 2f(1)\,\delta^{-1}\,|t_{kn}(x)|$$

□

Theorem 2.3. *For a sequence of modulus functions $F = (f_k)$ and any lacunary sequence $\theta = (k_r)$,*

$$[w_\sigma, F]_\theta \subset [w_\sigma^\infty, F]_\theta\,.$$

Proof. Let $F = (f_k)$ be a sequence of modulus functions and $x \in [w_\sigma, F]_\theta$. Put $\sup_k f_k(1) = M$. We can write

$$\frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x)|) \le \frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|) + \frac{1}{h_r}\sum_{k\in I_r} f_k(|l|)$$

$$\le \frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|) + T_l M$$

where T_l is integer number such that $|l| < T_l$. Hence $x \in [w_\sigma^\infty, F]_\theta$.

Now for any lacunary sequence $\theta = (k_r)$, we give connection between $[w_\sigma, F]_\theta$ and $[w_\sigma, F]$. □

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence with $\liminf_r q_r > 1$. Then for sequence of modulus functions $F = (f_k)$,*

$$[w_\sigma, F] \subset [w_\sigma F]_\theta$$

Proof. Suppose that $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r > 1 + \delta$ for all r. Then for $x \in [w_\sigma, F]$, we write

$$\frac{1}{k_r}\sum_{k=1}^{k_r} f_k(|t_{kn}(x-l)|) \ge \frac{1}{k_r}\sum_{k=1}^{k_r} f_k(|t_{kn}(x-l)|) + \frac{1}{k_r}\sum_{k=1}^{k_{r-1}} f_k(|t_{kn}(x-l)|)$$

$$= \frac{1}{k_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|)$$

$$\ge \frac{\delta}{1+\delta}\frac{1}{h_r}\sum_{k\in I_r} f_k(|t_{kn}(x-l)|)$$

By taking limit as $r \to \infty$ uniformly in, hence we obtain $x \in [w_\sigma, F]_\theta$. This completes the proof. □

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence with $\limsup q_r < \infty$. Then for any sequence of modulus functions $F = (f_k)$,*

$$[w_\sigma, F]_\theta \subset [w_\sigma, F]$$

Proof. If $\limsup q_r < \infty$, there exists $H > 0$ such that $q_r < H$ for all $r \ge 1$. Let $x \in [w_\sigma, F]_\theta$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \ge R$ and all n

$$A_j = \frac{1}{h_j}\sum_{k\in I_j} f_k(|t_{kn}(x-l)|) < \varepsilon.$$

We can also find $M > 0$ such that $A_j < K$ for all $j = 1, 2, \ldots.$ Now let m be any integer with $k_{r-1} < m \le k_r$, where $r > R$. We have

$$\frac{1}{m}\sum_{k=1}^{m} f_k(|t_{kn}(x-l)|) \le \frac{1}{k_{r-1}}\sum_{k=1}^{k_r} f_k(|t_{kn}(x-l)|)$$

$$= \frac{1}{k_{r-1}}\sum_{j=1}^{r}\sum_{k\in I_j} f_k(|t_{kn}(x-l)|)$$

$$= \frac{1}{k_{r-1}} \sum_{j=1}^{R} \sum_{k \in I_j} f_k \left(\left| t_{kn} \left(x - l \right) \right| \right) + \frac{1}{k_{r-1}} \sum_{j=R+1}^{r} \sum_{k \in I_j} f_k \left(\left| t_{kn} \left(x - l \right) \right| \right)$$

$$\leq \frac{1}{k_{r-1}} \left(\sup_{j \leq R} A_j \right) k_R + \frac{1}{k_{r-1}} \varepsilon \left(\sum_{j=R+1}^{r} h_j \right)$$

$$\leq \frac{1}{k_{r-1}} M k_R + \frac{1}{k_{r-1}} \varepsilon \left(h_{R+1} + h_{R+2} + ... + h_r \right)$$

$$\leq \frac{1}{k_{r-1}} M k_R + \varepsilon H$$

Since $k_{r-1} \to \infty$ as $r \to \infty$, it follows that

$$\frac{1}{m} \sum_{k=1}^{m} f_k \left(\left| t_{kn} \left(x - l \right) \right| \right) \to 0$$

uniformly in n and consequently $x \in [w_\sigma, F]$. Hence the proof completes. □

Theorem 2.6. *Let $\theta = (k_r)$ be a lacunary sequence 1$< \liminf_r q_r \leq \limsup q_r < \infty$. Then for any sequence of modulus functions $F = (f_k)$,*

$$[w_\sigma, F]_\theta = [w_\sigma, F]$$

Proof. Theorem 2.6 follows the theorems 2.5 and 2.4. □

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