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MINIMUM VALUE OF A MATRIX NORM WITH APPLICATIONS TO MAXIMUM PRINCIPLES FOR SECOND ORDER ELLIPTIC SYSTEMS

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Abstract. The purpose of this paper is to use an estimation of minimum value of a matrix norm to improve some results given by I.A.Rus in 1969, 1973, and A.S. Muresan in 1975.

1. **Introduction**

Let us consider the following operator:

$$Lu := \sum_{i,j=1}^{m} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} A_i \frac{\partial u}{\partial x_i} + A_0 u,$$

where $A_{ij}, A_i, A_0 \in C(\overline{\Omega}, M_n(\mathbb{R}))$ and $\Omega \subset \mathbb{R}^m$ is a bounded domain.

Let us also consider the following systems:

$$Lu = 0, \qquad (1)$$

$$Lu = f, \qquad (2)$$

where $f \in C(\overline{\Omega}, \mathbb{R}^n)$.

There are some maximum principles for the solutions of (1) (see for example [2], [5] and [8]).

In [5] the following principle is given:

Theorem 1. *Suppose that:*

1. *the system (1) is strongly elliptic,*
2. $e^* Le < 0$, *for each* $e \in C^2(\overline{\Omega}, \mathbb{R}^n)$, *with* $\|e\| := \left(\sum_{i=1}^{n} e_i^2 \right)^{1/2} = 1$.

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ *is a solution of (1), then* $\|u\| := \left(\sum_{i=1}^{n} u_i^2 \right)^{1/2}$ *attains his maximum value on* $\partial\Omega$.

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The aim of this paper is to find conditions which imply condition 2 of Theorem 1. This will be done in section 2 of this paper. In section 3 we shall try to improve some estimations for the norm of solution of system (2), estimations given in [4] and [6].

Let $A \in M_n(\mathbb{R}), J$ the Jordan normal form of A. We know that there exist a nonsingular matrix T such that $A = TJT^{-1}$.
We will denote:

$$\begin{aligned} \widetilde{\alpha} &= \begin{cases} \frac{1}{n}\sum\limits_{k=1}^{s} n_k\lambda_k, \lambda_k \in \mathbb{R} \\ \frac{1}{n}\sum\limits_{k=1}^{s} n_k \ Re\lambda_k, \lambda_k \in \mathbb{C}\backslash\mathbb{R} \end{cases} \\ \gamma_F &= \|T\|_F \cdot \left\|T^{-1}\right\|_F \\ m_F &= \|J - \widetilde{\alpha} I\|_F \end{aligned}$$

where λ_k are the eigenvalues of A, n_k is the number of λ_k which appears in Jordan blocks (generated by λ_k) and $\|\cdot\|_F$ is the euclidean norm of a matrix (see [1]).

We shall use the following result given in [1]:

Theorem 2. *Let $\varphi_{\|\cdot\|} : \mathbb{R} \to \mathbb{R}, \varphi_{\|\cdot\|}(\alpha) = \|A - \alpha I_n\|$, $\|\cdot\|$ being one of the following norms: $\|\cdot\|_F$, $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. In these conditions:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) \le \sqrt{n}\gamma_F m_F.$$

Remark 1. *In case of euclidean norm $\|\cdot\|_F$ and spectral norm $\|\cdot\|_2$ we have that $\varphi_{\|\cdot\|}(\widetilde{\alpha}) \le \gamma_F m_F$ (see [1]). Because $n \ge 2$, if $m_F \ne 0$, then:*

$$\varphi_{\|\cdot\|}(\widetilde{\alpha}) < \sqrt{n}\gamma_F m_F.$$

2. **Main result for the solution of system (1)**

In this section we shall give conditions under which condition 2 of Theorem 1 holds in case $A_{ij} = a_{ij} I_n$, $a_{ij} \in C(\overline{\Omega})$. Suppose that there exist $\delta > 0$ such that:

$$\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \ge \delta^2 \|\xi\|^2, \xi \in \mathbb{R}^n. \tag{3}$$

Theorem 3. *Suppose that (3) holds and:*

$$\xi^* A_0(x)\xi \le -\frac{1}{4\delta^2} n \|\xi\|^2 \sum_{i=1}^{m} (\gamma_F^i m_F^i)^2, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega. \tag{4}$$

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), u \ne 0$, is a solution of (1), then $\|u\| := \left(\sum\limits_{k=1}^{n} u_k^2\right)^{1/2}$ attains his maximum value on $\partial\Omega$.

Proof. Our result is based on the following remark which appears in [5]:

If, for each $x \in \Omega$, there exist $\widetilde{\alpha_i}(x) \in \mathbb{R}, i = \overline{1,m}$, such that:

$$\xi^* \begin{pmatrix} -a_{11}I_n & -a_{12}I_n & \dots & -a_{1m}I_n & 0 \\ -a_{21}I_n & -a_{22}I_n & \dots & -a_{2m}I_n & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -a_{m1}I_n & -a_{m2}I_n & \dots & -a_{mm}I_n & 0 \\ A_1(x) - \widetilde{\alpha_1}(x)I_n & A_2(x) - \widetilde{\alpha_2}(x)I_n & \dots & A_m(x) - \widetilde{\alpha_m}(x)I_n & A_0(x) \end{pmatrix} \xi < 0, \tag{5}$$

for all $\xi \in \mathbb{R}^{(m+1)n}, \xi \neq 0, \forall x \in \Omega$ then condition 2 of Theorem 1 holds.

So, it is enough to show that (4) implies (5).

Now it's easy to see that if, for each $x \in \Omega$, there exist $\varepsilon_i(x) > 0$ and $\widetilde{\alpha_i}(x) \in \mathbb{R}$, such that

$$\|A_i(x) - \widetilde{\alpha_i}(x)I_n\| < 2\varepsilon_i(x), i = \overline{1,m}, \tag{6}$$

$$\xi^* A_0(x)\xi \leq -\frac{1}{\delta^2}\|\xi\|^2 \sum_{i=1}^{m} \varepsilon_i^2(x), \forall \xi \in \mathbb{R}^n, \tag{7}$$

then (5) holds.

For simplicity we shall prove this in case $m = n = 2$.

We have:

$$\xi^* \begin{pmatrix} -a_{11}I_2 & -a_{12}I_2 & 0 \\ -a_{21}I_2 & -a_{22}I_2 & 0 \\ A_1(x) - \widetilde{\alpha_1}(x)I_2 & A_2(x) - \widetilde{\alpha_2}(x)I_2 & A_0(x) \end{pmatrix} \xi \leq -\delta^2(\xi_1^2 + \xi_3^2) -$$

$$-\delta^2(\xi_2^2 + \xi_4^2) + \delta^2(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) + \frac{1}{4\delta^2}\|\xi'\|^2 \|A_1(x) - \widetilde{\alpha_1}(x)I_2\|^2 +$$

$$+\frac{1}{4\delta^2}\|\xi'\|^2 \|A_2(x) - \widetilde{\alpha_2}(x)I_2\|^2 + \xi'^* A_0(x)\xi' <$$

$$< \frac{\varepsilon_1^2(x) + \varepsilon_2^2(x)}{\delta^2}\|\xi'\|^2 - \frac{\varepsilon_1^2(x) + \varepsilon_2^2(x)}{\delta^2}\|\xi'\|^2 = 0,$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^6, \xi \neq 0, \xi' = (\xi_5, \xi_6) \in \mathbb{R}^2, \xi' \neq 0$.

Now, according to Theorem 2 and Remark 1 we have that if $m_F^i \neq 0, i = \overline{1,m}$, then for each $x \in \Omega$, there exist $\widetilde{\alpha_i}(x) \in \mathbb{R}$ such that $\|A_i(x) - \widetilde{\alpha_i}(x)I_n\| < \sqrt{n}\gamma_F^i m_F^i$. So choosing $\varepsilon_i(x) = \frac{1}{2}\sqrt{n}\gamma_F^i m_F^i$, the proof is done.

Remark 2. *If $m_F^i = 0, i = \overline{1,m}$, then the conclusion of Theorem 3 holds if*

$$\xi^* A_0(x)\xi < 0, \forall \xi \in \mathbb{R}^n, \xi \neq 0, x \in \Omega$$

Example 1. *Let us consider the system (1) in case $m = n = 2$ with $A_1 = A_2 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$. We suppose that $a_2, a_3 > 0$. In this case we shall have: $\widetilde{\alpha}_1 = \widetilde{\alpha}_2 = a_1$, $\gamma_F^{A_1} = \gamma_F^{A_2} = \frac{a_2 + a_3}{\sqrt{a_2 a_3}}, m_F^{A_1} = m_F^{A_2} = \sqrt{2a_2a_3}, A_1 - a_1I_2 = A_2 - a_1I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}, \varepsilon_1 = \varepsilon_2 = a_2 + a_3$.*

The condition (4) becomes:

$$\xi^* A_0(x)\xi \leq -\frac{2}{\delta^2}(a_2 + a_3)^2 \|\xi\|^2, \xi \in \mathbb{R}^2, x \in \Omega. \tag{8}$$

If (3) and (8) holds, then we have:

$$\xi^* \begin{pmatrix} -a_{11}I_2 & -a_{12}I_2 & 0 \\ -a_{21}I_2 & -a_{22}I_2 & 0 \\ A_1 - a_1 I_2 & A_2 - a_2 I_2 & A_0(x) \end{pmatrix} \xi \leq \frac{1}{4\delta^2} \|\xi'\|^2 (a_2^2 + a_3^2) +$$

$$+\frac{1}{4\delta^2} \|\xi'\|^2 (a_2^2 + a_3^2) - \frac{2}{\delta^2}(a_2 + a_3)^2 \|\xi'\|^2 = \frac{1}{4\delta^2} \left[a_2^2 + a_3^2 - 4(a_2 + a_3)^2\right] \|\xi'\|^2 < 0,$$

where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \in \mathbb{R}^6, \xi \neq 0, \xi' = (\xi_5, \xi_6) \in \mathbb{R}^2, \xi' \neq 0.$

So, if (3) and (8) holds then, if $u \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2), u \neq 0,$ *is a solution of (1) in case* $m = n = 2$, *with* $A_1 = A_2 = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}, a_2, a_3 > 0,$ *then* $\|u\|$ *attains his maximum value on* $\partial\Omega$.

3. **Estimations for the solution of system (2)**

In this section we shall try to improve some estimation for the norm of the solution of system (2), estimations given in [4] and [6]. For other estimations see [3] and [8].

Theorem 4. *([4],[6]): Suppose that:*

1. *the system (2) is strongly elliptic,*
2. $e^* Le \leq -p^2$, *for each* $e \in C^2(\overline{\Omega}, \mathbb{R}^n)$, *with* $\|e\| = 1, p \in \mathbb{R}^*$.

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ *is a solution of (2), then:*

$$|u(x)| \leq \max\left\{\max_{x\in\partial\Omega} |u(x)|, \frac{1}{p^2} \max_{x\in\overline{\Omega}} |f(x)|\right\}, x \in \overline{\Omega}.$$

As in section 2, we shall try to find conditions under which condition 2 of Theorem 4 holds. In this way we shall be able to find a value of p.

In case m=1, system (2) becomes:

$$Ly := \frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)y = f(x), \tag{9}$$

where $B, C \in C([a, b], M_n(\mathbb{R})), f \in C([a, b], \mathbb{R}^n)$.

If $m_F \neq 0$, then we have the following result:

Theorem 5. *Suppose that:*

$$e^* C(x)e \leq -\frac{1}{4} n(\gamma_F m_F)^2, \tag{10}$$

$\forall e \in C^2([a,b],\mathbb{R}^n), \|e\| = 1, \forall x \in]a, b[.$

If $y \in C^2([a, b], \mathbb{R}^n), y \neq 0,$ *is a solution of (9), then:*

$$|y(x)| \leq \max\left\{|y(a)|, |y(b)|, \frac{4}{n\gamma_F^2 m_F^2 - \left\|B(x) - \widetilde{\beta}(x) I_n\right\|^2} \max_{x\in[a,b]} |f(x)|\right\}, x \in [a, b].$$

Proof. According to Theorem 2 and Remark 1 we have that, for each $x \in]a,b[$, there exist $\widetilde{\beta}(x) \in \mathbb{R}$ such that $\left\| B(x) - \widetilde{\beta}(x) I_n \right\| < \sqrt{n}\gamma_F m_F$.

We have:

$e^* L e = -\|e'\|^2 + e^* B(x) e' + e^* C(x) e = -\|e'\|^2 + e^* \left(B(x) - \widetilde{\beta}(x) I_n \right) e' + e^* C(x) e \leq -\|e'\|^2 + \left\| B(x) - \widetilde{\beta}(x) I_n \right\| \|e'\| + e^* C(x) e \leq \frac{1}{4} \left\| B(x) - \widetilde{\beta}(x) I_n \right\|^2 + e^* C(x) e \leq \frac{1}{4} \left\| B(x) - \widetilde{\beta}(x) I_n \right\|^2 - \frac{1}{4} n (\gamma_F m_F)^2 = -p^2(x) < 0.$

So $e^* L e \leq -p^2$ and hence and from Theorem 4, Theorem 5 is proved.

Remark 3. *In case that $m_F = 0$, if there exist $p \neq 0$ such that $e^* C(x) e \leq -p^2, \forall x \in]a,b[$, then the conclusion becomes:*

$$|y(x)| \leq \max \left\{ |y(a)|, |y(b)|, \frac{1}{p^2} \max_{x \in [a,b]} |f(x)| \right\}, x \in [a,b].$$

Example 2. *Let us consider the system (9) with $B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$ and $a_2, a_3 > 0$. In this case we shall have:*

$\widetilde{\beta} = a_1$, $\gamma_F = \frac{a_2 + a_3}{\sqrt{a_2 a_3}}, m_F = \sqrt{2 a_2 a_3}$ *, and* $B - \widetilde{\beta} I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}$. *The relation* (10), *becomes:*

$$e^* C(x) e \leq -(a_2 + a_3)^2, x \in]a,b[. \tag{11}$$

If (11) holds and $y \in C^2([a,b], \mathbb{R}^2)$ is a solution of (9), then:

$$|y(x)| \leq \max \left\{ |y(a)|, |y(b)|, \frac{4}{3a_2^2 + 8a_2 a_3 + 3a_3^2} \max_{x \in [a,b]} |f(x)| \right\}, x \in [a,b].$$

In case $m = 2$, $A_{ij} = I_n$, we shall consider the system:

$$Lu := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} + C(x,y) u = f(x,y), \tag{12}$$

where $A, B, C \in C(\overline{\Omega}, M_n(\mathbb{R})), f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $\Omega \subseteq \mathbb{R}^2$ is a bounded domain.

If $m_F^A \neq 0$, $m_F^B \neq 0$, then we have the following result:

Theorem 6. *Suppose that:*

$$e^* C(x,y) e \leq -\frac{1}{4} n \left[\left(\gamma_F^A m_F^A \right)^2 + \left(\gamma_F^B m_F^B \right)^2 \right], \tag{13}$$

$\forall e \in C^2(\overline{\Omega}, \mathbb{R}^n), \|e\| = 1, \forall (x,y) \in \Omega.$

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), u \neq 0$, is a solution of (12), then:

$$|u(x,y)| \leq \max \left\{ \max_{(x,y) \in \partial\Omega} |u(x,y)|, \frac{4}{p^2(x,y)} \max_{(x,y) \in \overline{\Omega}} |f(x,y)| \right\}, (x,y) \in \overline{\Omega},$$

where

$$p^2(x,y) = n \left(\gamma_F^A m_F^A \right)^2 + n \left(\gamma_F^B m_F^B \right)^2 - \|A(x,y) - \widetilde{\alpha}(x,y) I_n\|^2 - \left\| B(x,y) - \widetilde{\beta}(x,y) I_n \right\|^2.$$

Proof. According to Theorem 2 and Remark 1, if $m_F^A \neq 0, m_F^B \neq 0$, for each $(x,y) \in \Omega$, there exist $\widetilde{\alpha}(x,y), \widetilde{\beta}(x,y) \in \mathbb{R}$ such that:

$$\|A(x,y) - \widetilde{\alpha}(x,y)I_n\| < \sqrt{n}\gamma_F^A m_F^A$$

$$\left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\| < \sqrt{n}\gamma_F^B m_F^B$$

We have:

$$e^*Le = -\|e_x'\|^2 - \left\|e_y'\right\|^2 + e^*(A(x,y) - \widetilde{\alpha}(x,y)I_n)e_x' + e^*(B(x,y) - \widetilde{\beta}(x,y)I_n)e_y' +$$

$$+e^*C(x,y)e \leq \frac{1}{4}\|A(x,y) - \widetilde{\alpha}(x,y)I_n\|^2 + \frac{1}{4}\left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\|^2 + e^*C(x,y)e \leq$$

$$\leq \frac{1}{4}\left[\|A(x,y) - \widetilde{\alpha}(x,y)I_n\|^2 + \left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\|^2 - n\left(\gamma_F^A m_F^A\right)^2 - n\left(\gamma_F^B m_F^B\right)^2\right] =$$

$$= -p^2(x,y) < 0.$$

So $e^*Le \leq -p^2$ and hence and from Theorem 4, Theorem 6 is proved.

Remark 4. *In case that* $m_F^A = 0, m_F^B \neq 0$, *if* $e^*C(x,y)e \leq -\frac{1}{4}n\left(\gamma_F^B m_F^B\right)^2$, *then the conclusion holds with* $p^2(x,y) = n\left(\gamma_F^B m_F^B\right)^2 - \left\|B(x,y) - \widetilde{\beta}(x,y)I_n\right\|^2$.

Example 3. *Let us consider the system (12) with* $A = B = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_1 \end{pmatrix}$. *We suppose that* $a_2, a_3 > 0$. *In this case we shall have:* $\widetilde{\alpha} = \widetilde{\beta} = a_1$, $\gamma_F^A = \gamma_F^B = \frac{a_2+a_3}{\sqrt{a_2 a_3}}$, $m_F^A = m_F^B = \sqrt{2a_2a_3}$, $A - \widetilde{\alpha}I_2 = B - \widetilde{\beta}I_2 = \begin{pmatrix} 0 & a_2 \\ a_3 & 0 \end{pmatrix}$.

$$e^*C(x,y)e \leq -2(a_2+a_3)^2, \tag{14}$$

$$e^*Le \leq \frac{a_2^2 + a_3^2}{2} - 2(a_2+a_3)^2 < 0.$$

If (14) holds and $u \in C^2(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2), u \neq 0$, *is a solution of (12), we have:*

$$|u(x,y)| \leq \max\left\{\max_{(x,y)\in\partial\Omega}|u(x,y)|, \frac{2}{3a_2^2 + 8a_2a_3 + 3a_3^2}\max_{(x,y)\in\overline{\Omega}}|f(x,y)|\right\}, (x,y) \in \overline{\Omega}.$$

Let us consider now $A_{ij} = a_{ij}I_n$, $a_{ij} \in C(\overline{\Omega})$. System (2) becomes:

$$Lu := \sum_{i,j=1}^m a_{ij}(x)I_n\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^m A_i(x)\frac{\partial u}{\partial x_i} + A_0(x)u = f(x), \tag{15}$$

where $A_i, A_0 \in C(\overline{\Omega}, M_n(\mathbb{R})), f \in C(\overline{\Omega}, \mathbb{R}^n)$.

If $m_F^i \neq 0$, then we have the following result:

Theorem 7. *Suppose (3) holds and:*

$$e^*A_0(x)e \leq -\frac{1}{4\delta^2}n\sum_{i=1}^m(\gamma_F^i m_F^i)^2, \forall e \in C^2(\Omega, \mathbb{R}^n), \|e\| = 1, \forall x \in \Omega. \tag{16}$$

If $u \in C^2(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n), u \neq 0$, is a solution of (15), then:

$$|u(x)| \leq \max \left\{ \max_{x \in \partial\Omega} |u(x)|, \frac{4\delta^2}{n \sum_{i=1}^{m} (\gamma_F^i m_F^i)^2 - \sum_{i=1}^{m} \|A_i(x) - \widetilde{\alpha}_i(x) I_n\|^2} \max_{x \in \overline{\Omega}} |f(x)| \right\}, \ x \in \overline{\Omega}.$$

Remark 5. *In case that $m_F^i = 0$, if there exist $p \neq 0$ such that $e^* A_0(x) e \leq -p^2$, then:*

$$|u(x)| \leq \max \left\{ \max_{x \in \partial\Omega} |u(x)|, \frac{1}{p^2} \max_{x \in \overline{\Omega}} |f(x)| \right\}, x \in \overline{\Omega}.$$

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