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WEAKLY SINGULAR VOLTERRA AND FREDHOLM-VOLTERRA INTEGRAL EQUATIONS

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. Some existence and uniqueness theorems are established for weakly singular Volterra and Fredholm-Volterra integral equations in $C[a,b]$. Our method is based on fixed point theorems which are applied to the iterated operator and we apply the fiber Picard operator theorem to establish differentiability with respect to parameter. This method can be applied only for linear equations because otherwise we can't compute the iterated equation.

1. Introduction

The integral equation

$$u(x) = f(x) + \int_a^x K_1(x,s)u(s)ds, \tag{1}$$

with $f \in C[a,b]$ is weakly singular if there exists $L_1 \in C([a,b] \times [a,b])$ and $\alpha \in (0,1)$ such that $K_1(x,s) = \frac{L_1(x,s)}{|x-s|^\alpha}$ $\forall\ x,s \in [a,b]$ with $x \neq s$. In this case the kernel function K_1 is called *weakly singular*. The integral equation

$$u(x) = f(x) + \int_a^x K_1(x,s)u(s)ds + \int_a^b K_2(x,s)u(s)ds, \tag{2}$$

with $f \in C[a,b]$ is called weakly singular if at least one of the kernel functions K_1 and K_2 is weakly singular. In this paper we give an existence and uniqueness theorem for the equation 1 by using fixed point approach and we obtain the continuous dependence and differentiability with respect to a parameter. For equation 2 we study two different cases, in the first case K_1 is weakly singular and K_2 is continuous and in the second case both kernels are weakly singular. In both cases we obtain existence, uniqueness, continuous dependence and differentiability with respect to the parameter. We'll use the following theorems

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Theorem 1.1. *If (X,d) is a complete metric space and $T : X \to X$ is an operator with*

$$d(Tu,Tv) \leq L \cdot d(u,v)\ \forall\ u,v \in X,\ \text{where } 0 < L < 1,$$

then

1. *T has an unique fixed point u^*.*
2. *The sequence $u_{n+1} = Tu_n$, $\forall\ n \in \mathbb{N}$ is convergent to u^* for all $u_0 \in X$.*
3. *$d(u_n,u^*) \leq \frac{L^n}{1-L} \cdot d(u_1,u_0)\ \forall\ n \in \mathbb{N}$.*

Theorem 1.2. *(Fiber Picard operator's) [8] Let (V,d) be a generalized metric space with $d(v_1,v_2) \in \mathbb{R}_+^p$, and (W,ρ) a complete generalized metric space with $\rho(w_1,w_2) \in \mathbb{R}_+^m$. Let $A : V \times W \to V \times W$ be a continuous operator. If we suppose that:*

a) *$A(v,w) = (B(v), C(v,w))$ for all $v \in V$ and $w \in W$;*
b) *the operator $B : V \to V$ is a weakly Picard operator;*
c) *there exists a matrix $Q \in M_m(\mathbb{R}_+)$ convergent to zero, such that the operator $C(v,\cdot) : W \to W$ is a Q contraction for all $v \in V$,*

then the operator A is a weakly Picard operator. Moreover, if B is a Picard operator, then the operator A is a Picard operator.

Theorem 1.3. *If X is a set and $T : X \to X$ is a function such that the equation $T^n(u) = u$ has an unique solution u^*, than u^* is the unique solution of the equation $Tu = u$*

Theorem 1.4. *If (X,d) is a generalized complete metric space and $T : X \to X$ is an operator such that T^k is a contraction, then the sequence $u_{n+1} = Tu_n\ \forall\ n \in \mathbb{N}$ is convergent to the unique fixed point of T^k.*

In order to apply these theorems to weakly singular integral equations we need the following properties of the weakly singular kernels.

Theorem 1.5. *If $K(x,s) = \frac{L(x,s)}{|x-s|^\alpha}$ with $0 < \alpha < 1$ and $L \in C([a,b] \times [a,b])$, then the operator $T : C[a,b] \to C[a,b]$,*

$$(Tu)(x) = \int_a^x K(x,s)u(s)ds$$

is well defined ($Tu \in C[a,b]$).

Proof. If $a \leq x < x' \leq b$ and $\delta_1 > 0$ we have

$$|(Tu)(x') - (Tu)(x)| \leq \int_a^{x-\delta_1} |K(x',s) - K(x,s)||u(s)|ds +$$

$$+ \int_{x-\delta_1}^{x'-\delta_1} |K(x',s)||u(s)|ds + \int_{x-\delta_1}^{x} |K(x,s)||u(s)|ds +$$

$$+ \int_{x'-\delta_1}^{x'} |K(x',s)||u(s)|ds.$$

$u \in C[a,b]$, implies that there exists $M = \max\limits_{s\in[a,b]} |u(s)|$.

$K : [x - \frac{\delta_1}{2}, b] \times [a, x - \delta_1] \to \mathbb{R}$ is continuous so it is uniform continuous and $\forall\ \epsilon > 0$ there exists $\delta_2 > 0$ such that

$$|K(x', s) - K(x, s)| < \tfrac{\epsilon}{2M(b-a)} \text{ if } |x - x'| < \delta_2 \text{ and } s \leq x - \delta_1.$$

This implies

$$|(Tu)(x') - (Tu)(x)| \leq \frac{\epsilon}{2} + M \cdot \int\limits_{x-\delta_1}^{x'-\delta_1} |K(x', s)| ds +$$

$$+ M \cdot \int\limits_{x-\delta_1}^{x} |K(x, s)| + M \cdot \int\limits_{x'-\delta_1}^{x'} |K(x', s)| ds,$$

if $|x - x'| < \delta_2$. On the other hand we have the following inequalities:

$$\int\limits_{x-\delta_1}^{x'-\delta_1} |K(x', s)| ds \leq P \cdot \int\limits_{x-\delta_1}^{x'-\delta_1} \frac{ds}{(x' - s)^\alpha} = P \cdot \left(- \frac{(x' - s)^{1-\alpha}}{1 - \alpha} \Big|_{x-\delta}^{x'-\delta_1} \right)$$

$$= \frac{P}{1-\alpha} \left((x' - x + \delta_1)^{1-\alpha} - \delta_1^{1-\alpha}) \right) \leq \frac{P}{1-\alpha} \cdot (2(x' - x))^{1-\alpha} < \frac{\epsilon}{6M}$$

where $|x' - x| < \delta_3$, and $P = \max\limits_{x,s\in[a,b]} |L(x, s)|$.

$$\int\limits_{x-\delta_1}^{x} |K(x, s)| \leq P \cdot \int\limits_{x-\delta_1}^{x} \frac{ds}{(x - s)^\alpha} = \frac{P}{1-\alpha} \left(-(x - s)^{1-\alpha} \Big|_{x-\delta_1}^{x} \right) =$$

$$= \frac{P}{1-\alpha} \cdot \delta_1^{1-\alpha} < \frac{\epsilon}{6M}$$

for $\delta_1 \leq \delta_4$.

$$\int\limits_{x'-\delta_1}^{x'} |K(x', s)| \leq \frac{P}{1-\alpha} \delta_1^{1-\alpha} < \frac{\epsilon}{6M}$$

for $\delta_1 \leq \delta_3$. From these inequalities we deduce

$$|(Tu)(x') - (Tu)(x)| < \epsilon$$

if $|x - x'| < \min(\delta_1, \delta_2, \delta_3, \delta_4)$, so the operator T is well defined.

Theorem 1.6. *If K_1 or K_2 is weakly singular kernel, then the operator $T : C[a, b] \to C[a, b]$,*

$$(Tu)(x) = \int\limits_{a}^{x} K_1(x, s) u(s) ds + \int\limits_{a}^{b} K_2(x, s) u(s) ds$$

is well defined ($Tu \in C[a, b]$).

Proof. As in theorem 1.1 we can prove that the operator $T_2 : C[a,b] \to C[a,b]$,

$$(T_2u)(x) = \int_a^b K_2(x,s)u(s)ds$$

is well defined if K_2 is a weakly singular kernel, so T is well defined because it is the sum of two well defined operators .

Theorem 1.7. [6] *If K_1 and K_2 are weakly singular kernels and*

$$|K_1(x,s)| \leq \frac{P_1}{|x-s|_1^{\alpha}}, \quad |K_2(x,s)| \leq \frac{P_2}{|x-s|_2^{\alpha}},$$

where $P_1, P_2 \in \mathbb{R},\ 0 \leq \alpha_1 < 1,\ 0 \leq \alpha_2 < 1,$ then the function

$$K_3(x,s) = \int_a^b K_1(x,t)K_2(t,s)dt$$

satisfies the following conditions:

1. *If $\alpha_1 + \alpha_2 > 1$, the function $K_3(x,s)$ is a weakly singular kernel and*

$$|K_3(x,s)| < \frac{P_3}{|x-s|^{\alpha_1+\alpha_2-1}},$$

where $P_3 \in \mathbb{R}$.

2. *If $\alpha_1 + \alpha_2 = 1$, the function $K_3(x,s)$ is continuous for $x \neq s$ and*

$$|K_3(x,s)| < P_3 + P_4 \ln|x-s|,$$

where $P_3, P_4 \in \mathbb{R}$.

3. *If $\alpha_1 + \alpha_2 < 1$, the function $K_3(x,s)$ is continuous in $D = [a,b] \times [a,b]$.*

The proof can be found in [6] at pp. 374. An analogous theorem can be proved for the Volterra integral operator.

Theorem 1.8. *If the functions K_1 and K_2 are weakly singular kernels and*

$$|K_1(x,s)| \leq \frac{P_1}{(x-s)^{\alpha_1}},$$

$$|K_2(x,s)| \leq \frac{P_2}{(x-s)^{\alpha_2}},$$

for $x \geq s$, then the function

$$K_3(x,s) = \int_s^x K_1(x,t)K_2(t,s)dt$$

satisfies the following properties

1. *If $\alpha_1 + \alpha_2 > 1$, then K_3 is a weakly singular kernel and*

$$|K_3(x,s)| \leq \frac{P_3}{(x-s)^{\alpha_1+\alpha_2-1}}.$$

2. *If $\alpha_1 + \alpha_2 = 1$, then K_3 is continuous and $|K_3(x,s)| \leq P_4$.*

3. *If* $\alpha_1 + \alpha_2 < 1$*, then* K_3 *is continuous and*

$$|K_3(x,s)| \leq P_4 \cdot (x-s)^{1-\alpha_1-\alpha_2}.$$

2. The main results

2.1. The Volterra integral equation.

Theorem 2.1. *If* $K(x,s,\lambda) = \frac{L(x,s,\lambda)}{(x-s)^\alpha}$ *with* $L \in C([a,b] \times [a,b] \times [\lambda_1, \lambda_2])$ *and* $0 < \alpha < 1$*, then the equation*

$$u(x) = f(x) + \int\limits_a^x K(x,s,\lambda)u(s)ds \tag{3}$$

with $f \in C[a,b]$ *and* $\lambda \in [\lambda_1, \lambda_2]$ *has a unique solution in* $C([a,b])$ *and this solution can be obtained by successive approximation. This solution depends continuously on* λ *and if* K *is continuously differentiable with respect to* λ*, the solution is also continuously differentiable with respect to* λ*.*

Proof. Due to theorem 1.5 the operator

$$T : C[a,b] \to C[a,b],\ (Tu)(x) = f(x) + \int\limits_a^x K(x,s,\lambda)u(s)ds$$

is well defined. Theorem 1.8 implies that there exists $n \in \mathbb{N}^*$ such that the iterated kernel $K^{(n)}$ defined by the following relations $K^{(1)}(x,s,\lambda)=K(x,s,\lambda)$ and $K^{(j+1)}(x,s,\lambda) = \int\limits_s^x K(x,t,\lambda) \cdot K^{(j)}(t,s,\lambda)dt\ \forall\ j \geq 1$ is continuous. But any solution of the equation 3 satisfies the iterated equation

$$u(x) = f(x) + \sum_{i=1}^{n-1} \int\limits_a^x K^{(i)}(x,s,\lambda)f(s)ds + \int\limits_a^x K^{(n)}(x,s,\lambda)u(s)ds. \tag{4}$$

We apply theorem 1.1 to the operator $\bar{T} : C[a,b] \to C[a,b]$

$$(\bar{T}u)(x) = f(x) + \sum_{i=1}^{n-1} \int\limits_a^x K^{(i)}(x,s,\lambda)f(s)ds + \int\limits_a^x K^{(n)}(x,s,\lambda)u(s)ds. \tag{5}$$

which has a continuous kernel, so by choosing a Bielecki metric in $C[a,b]$ $\bar{T}$ is a contraction. This implies that the equation $\bar{T}u = u$ has an unique solution u^* in $C[a,b]$. By the other hand from theorem 1.3 we obtain that u^* is the unique solution of the equation $Tu = u$, because $\bar{T} = T^{(n)}$. From theorem 1.4 we deduce that the sequence of successive approximation $u_{n+1} = Tu_n$ is convergent to u^* for every $u_0 \in C[a,b]$. This implies that equation 3 has an unique continuous solution, and this can be approximated by successive approximation. By applying the same technique to the equation

$$u(x,\lambda) = f(x) + \int\limits_a^x K(x,s,\lambda)u(s,\lambda)ds \tag{6}$$

we obtain that u^* is the unique solution in $C([a,b]\times[\lambda_1,\lambda_2])$, so the solution is depending continuously on the parameter λ. To study the differentiability of the solution we apply theorem 1.2 with the following spaces and operators:

a) $V = C([a,b]\times[\lambda_1,\lambda_2])$ and $B = \bar{T}$

b) $W = C([a,b]\times[\lambda_1,\lambda_2])$ and

$$C(v,w)(x,\lambda) = g(x,\lambda) + \int_a^x K^{(n)}(x,s,\lambda)\cdot w(s,\lambda)ds + \int_a^x \frac{\partial K^{(n)}}{\partial\lambda}\cdot v(s,\lambda)ds$$

where $g(x,\lambda) = \sum_{i=1}^{n-1}\int_a^x \frac{\partial K^{(i)}(x,s)}{\partial\lambda} f(s)ds$

The operator $A = (B,C)$ satisfies the conditions of theorem 1.2 because in $C([a,b]\times[\lambda_1,\lambda_2])$ we use a Bielecki metric and $K^{(n)}$ is a continuous function. This implies the uniform convergence of the sequence $v_{n+1} = V(v_n)$ to the unique solution u^* of equation 6 and the uniform convergence of the sequence $w_{n+1} = C(v_n,w_n)$ to a function w^*. If we choose $v_0 \in C^1[a,b]\times[\lambda_1,\lambda_2]$ and $w_0 = \frac{\partial v_0}{\partial\lambda}$ due to the operator C (which was obtained by a formal differentiation of the operator B)we have $w_n = \frac{\partial v_n}{\partial\lambda}$ $\forall\ n\in\mathbb{N}$. The Weierstrass's theorem implies that w^* is continuous and $w^*(x,\lambda) = \frac{\partial u^*(x,\lambda)}{\partial\lambda}$. So the solution u^* is continuously differentiable with respect to the parameter λ.

Remark 2.1. *We can use a direct proof (without the iterated operators) if we use the following inequality:*

$$|Tu(x)-Tv(x)| \le \int_a^x \frac{\max\limits_{x,s\in[a,b],\lambda\in[\lambda_1,\lambda_2]}|L(x,s,\lambda)|}{|x-s|^\alpha}\cdot|u(s)-v(s)|ds \le$$

$$\le L^*||u-v||\cdot\int_a^x \frac{e^{\tau(s-a)}}{(x-s)^\alpha}ds \le \left(\int_a^x \frac{ds}{(x-s)^{\alpha p}}\right)^{\frac{1}{p}}\cdot\left(\int_a^x e^{\tau(s-a)q}ds\right)^{\frac{1}{q}} \le$$

$$\le \left(\frac{(b-a)^{1-\alpha\cdot p}}{1-\alpha\cdot p}\right)^{\frac{1}{p}}\cdot\frac{e^{\tau(x-a)}}{(\tau\cdot q)^{\frac{1}{q}}},$$

where $\alpha\cdot p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $L^* = \max\limits_{x,s\in[a,b],\lambda\in[\lambda_1,\lambda_2]}|L(x,s,\lambda)|$ *and*

$$||u-v|| = \max_{x\in[a,b],\lambda\in[\lambda_1,\lambda_2]}|u(x,\lambda)-v(x,\lambda)|\cdot e^{-\tau(x-a)}.$$

So we can choose τ such that the operator T be a contraction with the corresponding Bielecki metric.

2.2. **The Fredholm-Volterra integral equation.**

Theorem 2.2. *For the equation*

$$u(x) = f(x) + \int_a^x K_1(x,s,\lambda)y(s)ds + \int_a^b K_2(x,s,\lambda)y(s)ds \tag{7}$$

with

$$L_1 = \max_{x,s\in[a,b],\lambda\in[\lambda_1,\lambda_2]} |K_1(x,s,\lambda)|$$

and

$$L_2 = \frac{2 \cdot \max\limits_{x,s\in[a,b],\lambda\in[\lambda_1,\lambda_2]} |L(x,s,\lambda)|}{1-\alpha} \cdot (b-a)^{1-\alpha}$$

where $K_1, L \in C([a,b]\times[a,b]\times[\lambda_1,\lambda_2])$ *and* K_2 *is a weakly singular kernel* ($K_2(x,s,\lambda) = \frac{L(x,s,\lambda)}{|x-s|^\alpha}$, $0<\alpha<1$) *the iterated kernels are*

$$K_1^{(n+1)}(x,s,\lambda) = \int_s^x K_1(x,t,\lambda)K_1^{(n)}(t,s,\lambda)dt + \int_a^b K_2(x,t,\lambda)K_1^{(n)}(x,t,\lambda)dt \qquad (8)$$

and

$$K_2^{(n+1)}(x,s,\lambda) = \int_a^x K_1(x,t,\lambda)K_2^{(n)}(t,s,\lambda)dt + \int_a^b K_2(x,t,\lambda)K_2^{(n)}(x,t,\lambda)dt \qquad (9)$$

and the resolvent kernels are

$$R_1(x,s,\lambda) = \sum_{j=1}^{\infty} K_1^{(j)}(x,s,\lambda), \qquad (10)$$

$$R_2(x,s,\lambda) = \sum_{j=1}^{\infty} K_2^{(j)}(x,s,\lambda). \qquad (11)$$

If L_1 *and* L_2 *satisfies condition a) or b), there exist an unique continuous solution to the equation 7, this solution depends continuously on* λ *and if the functions* K_1 *and* L *are continuously differentiable with respect to* λ*, then the solution is also continuously differentiable with respect to* λ*. The solution of the equation 7 can be represented in the form*

$$u(x) = f(x) + \int_a^x R_1(x,s,\lambda)f(s)ds + \int_a^b R_2(x,s,\lambda)f(s)ds.$$

The series (10) and (11) are convergent if L_1 *and* L_2 *satisfy the condition a) or b)*

a) $\frac{L_1}{2-L_2(b-a)} + \left(e^{\frac{L_1(b-a)}{2-L_2(b-a)}} - 2\right) L_1L_2(b-a) < 0;$

b) $\frac{1}{b-a}\ln\frac{1-L_2(b-a)}{(b-a)^2L_1L_2} + \left(\frac{1-L_2(b-a)}{(b-a)^2}L_1L_2 - 2\right)(b-a)L_1L_2 > 0$ *and*

$$\frac{1}{b-a}\ln\frac{1-L_2(b-a)}{(b-a)^2L_1L_2}(1-L_2(b-a)) +$$

$$+(b-a)L_1L_2\left(2 - \frac{1-L_2(b-a)}{(b-a)^2L_1L_2}\right) - L_1 > 0.$$

Proof. Due to theorem 1.2 we can apply the same reasoning as in [1] theorem 2.2.

Remark 2.2. *By applying the Fiber Picard operator theorem ([8]) as in [1] we can prove that the solution is differentiable with respect to the parameter λ*

Theorem 2.3. *If in the equation 7 both kernels are singular,* $K_1(x,s,\lambda) = \frac{L_1^*(x,s,\lambda)}{|x-s|^{\alpha_1}}$ *and* $K_2(x,s,\lambda) = \frac{L_2^*(x,s,\lambda)}{|x-s|^{\alpha_2}}$ *with* $L_1^*, L_2^* \in C([a,b]\times[a,b]\times[\lambda_1,\lambda_2])$*,* $0<\alpha_1<1$*,* $0<\alpha_2<1$ *and the numbers*

$$L_1 = \max_{x,s\in[a,b],\lambda\in[\lambda_1,\lambda_2]} |K_1^{(n)}(x,s,\lambda)| \tag{12}$$

and

$$L_2 = \max_{x,s\in[a,b],\lambda\in[\lambda_1,\lambda_2]} |K_2^{(n)}(x,s,\lambda)| \tag{13}$$

satisfies condition a) or b) from theorem 2.2 then equation 7 has an unique solution in $C[a,b]\times[\lambda_1,\lambda 2]$*. If in addition the functions* L_1^* *and* L_2^* *are continuously differentiable with respect to the parameter* λ*, the solution is also continuously differentiable with respect to* λ*.*

Proof. The iterated equation is

$$u(x) = f(x) + \sum_{j=1}^{n-1}\int_a^x K_1^{(j)}(x,s,\lambda)\cdot f(s)ds + \sum_{j=1}^{n-1}\int_a^b K_2^{(j)}(x,s,\lambda)\cdot f(s)ds +$$

$$+\int_a^x K_1^{(n)}(x,s,\lambda)u(s)ds + \int_a^b K_1^{(n)}(x,s,\lambda)u(s)ds$$

where the iterated kernels are defined by the relations 8 and 9. Due to theorem 1.5 and 1.6 the function

$$g_1(x,\lambda) = f(x) + \sum_{j=1}^{n-1}\int_a^x K_1^{(j)}(x,s,\lambda)\cdot f(s)ds + \sum_{j=1}^{n-1}\int_a^b K_2^{(j)}(x,s,\lambda)\cdot f(s)ds$$

is a continuous function. From theorem 1.7 and 1.8 we deduce that if $\max(\alpha_1,\alpha_2) < \frac{n-1}{n}$ and $\max\left(\frac{\alpha_2}{1-\alpha_1}, \frac{\alpha_1}{1-\alpha_2}\right) < n$ than $K_1^{(n)}$ and $K_2^{(n)}$ are continuous kernels so we can apply theorem 1.2 from [1] (because L_1 and L_2 satisfy a) or b)). From this theorem we deduce that the equation 7 has an unique solution u^* in $C[a,b]\times[\lambda_1,\lambda_2]$. This u^* is also the unique solution of the equation 7 because of theorem 1.3 and can be approximated by successive approximation due to theorem 1.4. To study the differentiability of the solution we apply theorem 1.2 again with the following spaces and operators:

a) $V = C([a,b]\times[\lambda_1,\lambda_2])$ and

$$(Bu)(x) = g_1(x,\lambda) + \int_a^x K_1^{(n)}(x,s,\lambda)u(s)ds + \int_a^b K_1^{(n)}(x,s,\lambda)u(s)ds$$

b) $W = C([a,b] \times [\lambda_1, \lambda_2])$ and

$$C(v,w)(x,\lambda) = \frac{\partial g_1(x,\lambda)}{\partial \lambda} + \int_a^x K_1^{(n)}(x,s,\lambda) \cdot w(s,\lambda)ds +$$

$$+ \int_a^x \frac{\partial K_1^{(n)}}{\partial \lambda} \cdot v(s,\lambda)ds + \int_a^b K_2^{(n)}(x,s,\lambda) \cdot w(s,\lambda)ds + \int_a^b \frac{\partial K_2^{(n)}}{\partial \lambda} \cdot v(s,\lambda)ds,$$

where $\frac{\partial g_1(x,\lambda)}{\partial \lambda} = \sum_{j=1}^{n-1} \int_a^x \frac{\partial K_1^{(j)}(x,s,\lambda)}{\partial \lambda} \cdot f(s)ds + \sum_{j=1}^{n-1} \int_a^b \frac{\partial K_2^{(j)}(x,s,\lambda)}{\partial \lambda} \cdot f(s)ds$

The operator $A = (B, C)$ satisfies the conditions of theorem 1.2 because in $C([a,b] \times [\lambda_1, \lambda_2])$ we use a Bielecki metric and $K^{(n)}$ is a continuous function. This implies the uniform convergence of the sequence $v_{n+1} = V(v_n)$ to the unique solution u^* of equation 7 and the uniform convergence of the sequence $w_{n+1} = C(v_n, w_n)$ to a function w^*. If we choose $v_0 \in C^1[a,b] \times [\lambda_1, \lambda_2]$ and $w_0 = \frac{\partial v_0}{\partial \lambda}$ due to the operator C (which was obtained by a formal differentiation of the operator B)we have $w_n = \frac{\partial v_n}{\partial \lambda}$ $\forall\, n \in \mathbb{N}$. The Weierstrass's theorem implies that w^* is continuous and $w^*(x,\lambda) = \frac{\partial u^*(x,\lambda)}{\partial \lambda}$. So the solution u^* is continuously differentiable with respect to the parameter λ.

Remark 2.3. 1. *Conditions 12 and 13 can be transferred inductively to the original kernels, but the conditions obtained are much more technical.*

2. *By using the same inequalities as in remark 2.1 we can avoid the use of the iterated kernels to obtain existence and uniqueness.*

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