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MAXIMAL FIXED POINT STRUCTURES

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. Examples, counterexamples and properties of the maximal fixed point structures are given.

1. **Introduction**

Let X be a nonempty set and $P(X) := \{Y \subseteq X \mid Y \neq \emptyset\}$. For $A, B \in P(X)$ we denote
$\mathbb{M}(A, B) := \{f : A \to B \mid F \text{ is an operator}\}$, $\mathbb{M}(A) := \mathbb{M}(A, A)$.

Definition 1.1. *(Rus* [39], [40], [41]*). A triple* $(X, S(X), M)$ *is a fixed point structure (briefly* FPS*) iff*
(i) $S(X) \subseteq P(X)$, $S(X) \neq \emptyset$;
(ii) M *is an operator which attaches to each pair* (A, B), $A, B \in P(X)$, *a nonempty subset of* $\mathbb{M}(A, B)$ *such that, for any* $Y \in P(X)$, *if* $Z \subseteq Y$, $Z \neq \emptyset$, $f(Z) \subseteq Z$, *then* $f|_Z \in M(Z)$, *for all* $f \in M(Y)$;
(iii) every $Y \in S(X)$ *has the fixed point property (briefly* FPP*) with respect to* $M(Y)$.

Definition 1.2. *(Rus* [43]*). The triple* $(X, S(X), M)$ *which satisfies* (i) *and* (iii) *in Definition 1.1 is called weak fixed point structure (briefly* $WFPS$*).*

Let $(X, S(X), M)$ be a FPS and $S_1(X) \subseteq P(X)$ such that $S_1(X) \subseteq S(X)$.

Definition 1.3. *(Rus [45]). The* FPS $(X, S(X), M)$ *is maximal in* $S_1(X)$ *iff we have*
$S(X) = \{A \in S_1(X) \mid f \in M(A) \text{ implies that } F_f \neq \emptyset\}$.

The aim of this paper is to give some examples of maximal FPS and to study the maximal FPS. Some open problems are formulated. Throughout the paper we follow terminologies and notations in [45] (see also [41], [42]).

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2. Examples and counterexamples

Example 2.1. *The trivial FPS is maximal in $P(X)$. In this case X is a nonempty set, $S(X) := \{\{x\} \mid x \in X\}$ and $M(Y) := \mathbb{M}(Y)$. We remark that if $cardY \geq 2$ there exists an operator $f : Y \to Y$ such that $F_f = \emptyset$.*

Example 2.2. *The Tarski FPS isn't maximal in $P(X)$. In this case $(X, \leq)$ is a partial ordered set, $S(X) :=\{ Y \in P(X) \mid (Y, \leq)$ is a complete lattice $\}$ and $M(Y) := \{f : Y \to Y \mid f$ is an increasing operator$\}$. To prove this assertion we consider $X := \mathbb{R}^2$ which is partial ordered by*

$$(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2.$$

We consider $Y = \{(1, 1), (1, 5), (2, 4)\}$ and we remark that $(Y, \leq)$ has the FPP with respect to increasing operators but $(Y, \leq)$ isn't a lattice.

Remark 2.1. *For other results see:* [7], [30], [30], [34], [46].

Example 2.3. *The Tarski FPS, $(X, S(X), M)$ is maximal in $S_1(X)$, for all ordered set $(X, \leq)$, where $S_1(X) := \{Y \in P(X) \mid (Y, \leq)$ is a lattice$\}$.By a theorem of Davies (*[14], [34]*) it follows that $Y \in S(X)$.*

Example 2.4. *The Schauder FPS isn't, in general, maximal in $P(X)$. In this example X is a Banach space, $S(X) := P_{cp,cv}(X)$ and $M(A, B) := C(A, B)$. For $Y \notin P_{cp,cv}(X)$ with topological FPP see* [4], [18], [24], [35] *and* [37].

We have

Theorem 2.1. *The Schauder FPS is maximal in $P_{b,cl,cv}(X)$.*

3. FPS of contractions

Let (X, d) be a complete metric space, $S(X) := P_{cl}(X)$ and $M(Y) := \{f : Y \to Y \mid f$ is a contraction$\}$. By definition $(X, S(X), M)$ is the FPS of contractions. It is clear that the FPS of contractions is maximal iff

$$(Y \in P(X), f \in M(Y) \Rightarrow F_f \neq \emptyset) \Rightarrow Y \in P_{cl}(X).$$

This problem is studied by M-C. Anisiu and V. Anisiu [6]. The main results are the following

Theorem 3.1. *(*[6], [12]*) There exists a complete metric space and a nonclosed subset with FPP with respect to contractions.*

Theorem 3.2. *(*[6]*) Let X be a Banach space and $Y \in P(X)$ a convex set with $IntY \neq \emptyset$. If each contraction $f : Y \to Y$ has a fixed point, then Y is closed.*

Remark 3.1. *For other results see* [13], [22], [26], [28].

4. Some properties of the maximal FPS

Let $\mathcal{C}$ be the class of structured sets (the class of sets, the class of all partial ordered sets, the class of Banach spaces, the class of Hausdorff topological spaces,...). Let S be an operator which attaches to each $X \in \mathcal{C}$ a nonempty set set $S(X) \subseteq$

$P(X)$. By M we denote an operator which attaches to each pair (A,B), $A \in P(X)$, $B \in P(Y)$, $X, Y \in \mathcal{C}$, a subset $M(A,B) \subseteq \mathbb{M}(A,B)$.

We have

Lemma 4.1. *Let $X \in \mathcal{C}$ and $(X, S(X), M)$ be a maximal FPS. Let $A \in S(X)$ and $B \in P(A)$. If there exists a retraction $r \in M(A,B)$ of A onto B such that*

$$f \in M(B) \Rightarrow f \circ r \in A$$

then $B \in S(X)$.

Proof. Let $f \in M(B)$. Then $f \circ r \in M(A)$. ¿From $A \in S(X)$ it follows that $F_{f \circ r} \neq \emptyset$. Let $x^* \in F_{f \circ r}$. We have $f(r(x^*)) = x^*$. We remark that $x^* \in B$ and so we have $f(x^*) = x^*$. By the maximality of $(X, S(X), M)$ it follows that $B \in S(X)$. □

Lemma 4.2. *Let $X, Y \in \mathcal{C}$. Let $(X, S(X), M)$ and $(Y, S(Y), M)$ be two FPS. Let $A \in S(X)$ and $B \in S(Y)$. We suppose that:*
i) $(Y, S(Y), M)$ is a maximal FPS;
ii) there exists a bijection $\varphi \in M(A,B)$ such that $\varphi^{-1} \circ g \circ \varphi \in M(A)$, for all $g \in M(B)$.
Then $B \in S(Y)$.

Proof. Let $f \in M(B)$. Then, from ii), it follows that $F_{\varphi^{-1} \circ f \circ \varphi} \neq \emptyset$. Let $x^* \in F_{\varphi^{-1} \circ f \circ \varphi}$. We remark that $\varphi(x^*) \in F_f$. So, by the maximality of $(Y, S(Y), M)$, we have $B \in S(Y)$. □

5. Open problems

The above considerations give rise to the following open problems.

Problem 1 Characterize the partial ordered sets with FPP with respect to increasing operator.

References: K. Baclavski and A. Bjőrner [7], A.C. Davies [14], G. Markowsky [32], J.D. Mashburn [33], I.A Rus [34], L.E. Ward [46].

Problem 2. Characterize the metric space with the FPP with respect to isometric operators.

References: K. Goebel and W.A. Kirk [20], W.A. Kirk and B. Sims [28], A.T.-M. Lau [29].

Problem 3.Characterise the metric space with the FPP with respect to contractions.

References: R.P. Agarwal, M. Meehan and D.O'Regan [4], M.C. Anisiu and V. Anisiu [6], V. Conserva and S. Rizzo [13], T.K. Hu [22], J. Jachymski [26], W.A. Kirk and B. Sims [28], I.A.Rus [45], H. Cohen [12].

Problem 4. Characterize the topological spaces with FPP with respect to continuous operators.

References: V.N. Akis [5], R.F. Brown [9], E.H. Connel [12], J. Dugundji and A.

Granas [18], A.A. Fora [19], W. Hans [21], S.Y. Husseini [23], E. de Pascale, G. Trombetta and H. Weber [16], I.A. Rus [35], [37].

Problem 5. Characterize the categories (S. MacLane [31]) with the FPP (I.A.Rus [38]).

References: J. Adàmek and V. Koubek [1], J. Adàmek, V. Koubek and J. Reitermann [2], A. Bjōrner [8], J. Isbel and B. Mitchel [25], J. Lambek [30], I.A. Rus [34], [38] and [43].

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