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ON THE CONVERGENCE OF THE SOLUTION OF THE QUASI-STATIC CONTACT PROBLEMS WITH FRICTION USING THE UZAWA TYPE ALGORITHM

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

Abstract. The aim of the paper is to prove the convergence of a Uzawa type algorithm for a dual mixed variational formulation of a quasi-static contact problem with friction. This problem is considered as a saddle point problem which is approximated with the mixed finite element, where the stress, displacement and tangential displacement on the contact boundary will be simultaneously computed.

1. Introduction

The quasi-static model of the contact problems with friction, without the inertia effects, was proposed by [14] and consists of the formulation obtained through the approximation with finite differences of the variational inequality. The proof of the existence and uniqueness is based on the hypothesis that the displacements satisfy some conditions of regularity and the friction coefficient is small enough. The static contact problem with friction cannot describe the evolutive state of the contact conditions. For of this reason, the quasi-static formulation, of the contact problem with friction is preferred, which contains a dynamic formulation of the contact conditions and the inertial term is no longer used. Through the temporal discretization of the quasi-static contact problem, the so called incremental problem is obtained, equivalent with a sequence of static contact problems. Therefore, the quasi-static problem is solved step by step, at each time small deformations and displacements are calculated and are added at those calculated previously, as a result of a few small modifications of the applied forces, of the contact zone and of the contact conditions. Although, at each increment the dependence of the load-way is neglected, this hypothesis takes into account the way the applied forces change (modify themselves). From a mathematical point view, the problem obtained at each step is similar with a static problem.

This dual mixed variational formulation problem is descretized by the mixed finite element method and an Uzawa type algorithm is proposed. The iterative formulation of this algorithm is deduced and its convergence is proved.

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The existence of solutions for the discrete problem by the mixed element method was obtained by Haslinger [7]. The contact problem has been recently studied by Andersen [11] and Rocca and Cocou [6] who proved that there exists a solution if the friction coefficient is small enough, and smooth and the contact functional is regular.

In this article is assumed that normal component of the stress vector and the contact zone is known.

2. **Classical and variational formulation**

Let $\Omega \subset \mathbb{R}^d, d = 2$ or 3, the polygonal domain occupied by a linear elastic body, and its boundary is denoted by Γ. Let Γ_1, Γ_2 and Γ_c be three open disjoint parts of Γ such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_c$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_c = \varnothing$ and mes $(\Gamma_1) > 0$. We assume for the simplicity that Γ_c is a segment for $d = 2$ and a polygon for $d = 3$. We denote by $\boldsymbol{u} = (u_1, \ldots, u_d)$ the displacement field, $\boldsymbol{\varepsilon} = (\varepsilon_{ij}(\mathbf{u})) = \left(\frac{1}{2}(u_{i,j} + u_{j,i})\right)$ the strain tensor and $\boldsymbol{\sigma} = (\sigma_{ij}(\mathbf{u})) = (a_{ijkl}\varepsilon_{kl}(\mathbf{u}))$ the stress tensor with the usual summation convention, where $i, j, k, l = 1, \ldots, d$. For the normal and tangential components of the displacement vector and stress vector, we use the following notation: $\mathbf{u}_N = u_i \cdot n_i$, $\boldsymbol{u}_T = \boldsymbol{u} - \boldsymbol{u}_N \cdot \mathbf{n}$, $\boldsymbol{\sigma}_N = \boldsymbol{\sigma}_{ij} u_i n_j$, $(\boldsymbol{\sigma}_T)_i = \boldsymbol{\sigma}_{ij} n_j - \boldsymbol{\sigma}_N \cdot n_i$, where $\mathbf{n} = (n_i)$ is the outward unit normal vector to $\partial\Omega$.

Lets us denote by $\boldsymbol{f}$ and $\boldsymbol{h}$ the density of body forces and traction forces, respectively. We assume that $a_{ijkl} \in L^\infty(\Omega)$, $l \leq i, j, k, l \leq d$, with usual condition of symmetry and elasticity, that is

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad l < i, j, k, l \leq d$$

$$\exists\, m_0 > 0,\ \forall\, \xi = (\xi_{ij}) \in \mathbb{R}^{d^2},\ \xi_{ij} = \xi_{ji},\ l \leq i,\ j \leq d,\ a_{ijkl}\, \xi_{ij}\, \xi_{kl} \geq m_0 |\xi|^2 .$$

In this conditions, the fourth-order tensor $\boldsymbol{a} = (a_{ijkl})$ is invertible a.e. on Ω and we denote its inverse $\boldsymbol{b} = (b_{ijkl})$, and $\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})) = (b_{ijkl}\sigma_{kl}(\boldsymbol{u}))$, $i, j, k, l = 1, \ldots, d$.

The classical contact problem with dry friction in elasticity is which the normal stress $\sigma_N(u)$ and Γ_c is assumed known, is follows: Find $\boldsymbol{u} = \boldsymbol{u}(x, t)$ such that $\boldsymbol{u}(0, \cdot) = \boldsymbol{u}^0(\cdot)$ in Ω and all $t \in [0, T]$,

$$-\mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega \tag{2.1}$$

$$\boldsymbol{\sigma}_{ij}(\boldsymbol{u}) = a_{ijkl} \cdot \varepsilon_{kl}(\boldsymbol{u}), \quad \text{in } \Omega \tag{2.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_1 \tag{2.3}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{h} \quad \text{on } \Gamma_2 \tag{2.4}$$

$$u_N \leq 0,\ \boldsymbol{\sigma}_N(u) \leq 0,\ u_N \boldsymbol{\sigma}_N(u) = 0 \quad \text{on } \Gamma_c \tag{2.5}$$

$$\mu_F |\boldsymbol{\sigma}_N(\boldsymbol{u})| = t, \quad t > 0$$

$$|\boldsymbol{\sigma}_T| < t \Rightarrow \dot{u}_T = 0;\ |\boldsymbol{\sigma}_T| = t \Rightarrow \exists\, \lambda \geq 0, \text{ s.t. } \dot{u}_T = -\lambda \boldsymbol{\sigma}_T \text{ on } \Gamma_c \tag{2.6}$$

where $\boldsymbol{u}^0$ is denoted the initial displacement of the body.

Condition (2.6) defines a form of Coulomb's law of friction for elastostatic problems: μ_F is the coefficient of friction $\mu_F \in L^\infty(\Gamma_c)$, $\mu_F \geq \mu_0$ a.e. on Γ_c.

The dual mixed variational formulation of the (2.1) - (2.6) in which stress, displacement and tangential displacement on contact zone are considerate unknown, it is shown the saddle-point problem with the form:

Find $(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\lambda}) \in S_t \times V \times \Lambda$ for all $t \in [0, T]$, such that

$$L(\boldsymbol{\sigma}, \boldsymbol{v}, \boldsymbol{\mu}) \leq L(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\lambda}) \leq L(\boldsymbol{\tau}, \boldsymbol{u}, \boldsymbol{\lambda}) \; \forall \; (\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\mu}) \in S_0 \times V \times \Lambda, \tag{2.7}$$

where $\boldsymbol{u} \in W^{1,2}(0, T; V)$, $\boldsymbol{\sigma} \in W^{1,2}(0, T; \mathcal{S})$, $\boldsymbol{f} \in W^{1,2}(0, T; [L^2(\Omega)]^d)$, $\boldsymbol{h} \in W^{1,2}(0, T; [L^2(\Gamma)]^d)$ with $supp(h(t)) \subset \Gamma_2$ for all $t \in [0, T]$.

$$L(\boldsymbol{\tau}, \boldsymbol{v}, \boldsymbol{\mu}) = J_0(\boldsymbol{\tau}) - (\operatorname{div} \boldsymbol{\tau}, \dot{\boldsymbol{v}}) - < \boldsymbol{t}, \; \boldsymbol{\mu} >_{\Gamma_c} \tag{2.8}$$

$$J_0(\boldsymbol{\tau}) = \frac{1}{2} \, a^*(\boldsymbol{\tau}, \boldsymbol{\tau}) + (\boldsymbol{f}, \operatorname{div} \boldsymbol{\sigma} + \dot{\boldsymbol{u}}) \tag{2.9}$$

$$\boldsymbol{t} = \mu_F |\boldsymbol{\sigma}_N(\boldsymbol{u})|, \quad \text{and} \quad \boldsymbol{\mu} = |\boldsymbol{u}_T| \text{ on } \Gamma_c \tag{2.10}$$

$$S_0 = \Big\{ \boldsymbol{\tau} | \tau_{ij}, \tau_{ij,j} \in L^2(\Omega), \; \tau_{ij} = \tau_{ji}, \; \boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \text{ a.e. on } \Gamma_2^f \Big\} \tag{2.11}$$

$$S_t = \big\{ \boldsymbol{\tau} | \tau_{ij}, \tau_{ij,j} \in L^2(\Omega), \; \tau_{ij} = \tau_{ji}, \; \boldsymbol{\tau} \cdot \boldsymbol{n} = t \text{ a.e. on } \Gamma_2 \big\} \tag{2.12}$$

$$S = \big\{ \boldsymbol{\tau} | \tau_{ij} \in L^2(\Omega), \; \tau_{ij} = \tau_{ji}, \; \tau_{ij,j} \in L^2(\Omega) \big\}$$

endowed with inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{S}} = \int_\Omega \sigma_{ij} \, \tau_{ij} \, dx. \tag{2.13}$$

Norm $\| \cdot \|_{\mathcal{S}}$ is then

$$\|\boldsymbol{\tau}\|_{\mathcal{S}} = (\boldsymbol{\tau}, \boldsymbol{\tau})_{\mathcal{S}}^{1/2} \tag{2.14}$$

$$\text{and} \qquad a^*(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega b_{ijkl} \, \sigma_{kl} \, dx \, . \tag{2.15}$$

Γ_2^f can be regarded as part of Γ_2 where $h \equiv 0$,

$$\Lambda = \{ \boldsymbol{\mu} \in H_{00}^{1/2}(\Gamma_c) | \, \boldsymbol{\mu} \geq 0 \text{ on } \Gamma_2^f \} \tag{2.16}$$

$$V = \{ \boldsymbol{v} \in H^1(\Omega) | \, \boldsymbol{v} / \Gamma_1 = 0 \} \tag{2.17}$$

$$H_{00}^{1/2}(\Gamma_c) = \{ \boldsymbol{\mu} \in H^{1/2}(\Gamma_c) | \rho^{-1/2} \boldsymbol{\mu} \in L^2(\Gamma_c) \} \, . \tag{2.18}$$

The norm of $H_{00}^{1/2}(\Gamma_c)$ is defined by

$${}_{00}\|\boldsymbol{\mu}\|_{1/2,\Gamma_c} = \Big\{ \|\boldsymbol{\mu}\|_{1/2,\Gamma_c}^2 + \|d^{-1/2} \boldsymbol{\mu}\|_{0,\Gamma_c}^2 \Big\}^{1/2} \quad , \tag{2.19}$$

where d denotes the distance between the point on Γ_c and the end point of Γ_c see [4].

3. The time discretisation and the mixed finite element approximation of the saddle point problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded and $(T_h)_h$ a triangulation of Ω. We assume that each triangulation is compatible with the partition of Γ. i.e. each point where the boundary condition changes is a node of a set Ω_i, where $\overline{\Omega} = \cup_{i\in J_h}\overline{\Omega}_i$, with $\Omega_k \cup \Omega_l = \varnothing$ for all $k, l \in J_h, k \neq l$.

The finite element approximation to the saddle-point problem (2.7) is as follow:

Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h) \in S_t^h \times V_h \times \Lambda_h$ for all $t \in [0, T]$, such that

$$L(\boldsymbol{\sigma}_h, \boldsymbol{v}_h, \boldsymbol{\mu}_h) \leq L(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h) \leq L(\boldsymbol{\tau}_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h),\ \forall\ (\boldsymbol{\tau}_h, \boldsymbol{v}_h, \boldsymbol{\mu}_h) \in S_0^h \times V_h \times \Lambda_h \tag{3.1}$$

where $S_0^h = S_0 \cap S_h$, $S_t^h = S_h$, $\Lambda_h = M_h \cap \Lambda$ and S_h, V_h, M_h are subspaces of finite elements of S, V and $H_{00}^{1/2}(\Gamma_c)$, respectively. Let S_h be RT_1, Raviart-Thomas space, V_h the space of the piecewise constant and M_h piecewise continuous linear subspace of $H_{00}^{1/2}(\Gamma_c)$, is called the mortar space [10], as well.

We assume that the initial displacement field u satisfies the compatibility conditions, see ([8]).

The discrete Babuška-Brezzi condition should be satisfied for the dual mixed finite element method. It means to find an interpolation operator π_h from $\boldsymbol{S}$ to Ω^h, such that:

$$b(\boldsymbol{\tau} - \pi_h\boldsymbol{\tau}, \boldsymbol{v}_h, \boldsymbol{\lambda}_h) = 0 \tag{3.2}$$

$$\|\pi_h\boldsymbol{\tau}\|_s \leq c\|\boldsymbol{\tau}\|_s, \quad \forall\, \boldsymbol{\tau} \in S, \tag{3.3}$$

that means, for all $\pi_h\boldsymbol{\tau} \in S_h$ we have

$$\int_\Omega \operatorname{div}(\boldsymbol{\tau} - \pi_h\boldsymbol{\tau})\boldsymbol{v}_h dx + \int_{\Gamma_c} (\boldsymbol{\tau}_N - \pi_h\boldsymbol{\tau}_N)\boldsymbol{\mu}_h ds = 0, \quad (\forall\, \boldsymbol{v}_h \in V_h, \boldsymbol{\mu}_h \in \Lambda_h). \tag{3.4}$$

Let

$$\int_\Omega \operatorname{div}(\boldsymbol{\tau} - \pi_h\boldsymbol{\tau})\boldsymbol{v}_h dx = 0, \quad (\forall\, \boldsymbol{v}_h \in V_h) \tag{3.5}$$

$$\int_{\Gamma_c} (\boldsymbol{\tau}_N - (\pi_h\boldsymbol{\tau}_h)_N\boldsymbol{\mu}_h ds = 0, \quad \forall\, \boldsymbol{\mu}_h \in \Lambda_h\,. \tag{3.6}$$

Because $\boldsymbol{\sigma}_N(\boldsymbol{u})$ on Γ_c is regarded as given, applying Green's formula to equation (3.5) in the finite element discrete form, is clear that the elements of subspace S_h satisfies (3.2) and (3.3) and we finally obtain further

$$\|\boldsymbol{\tau}_{Nh}\|_{0,\Gamma_c} \leq \|\boldsymbol{\tau}_h\|_{0,\Omega} \leq \|\boldsymbol{\tau}_h\|_S, \quad (\forall\, \boldsymbol{\tau}_h \in \boldsymbol{S}_h). \tag{3.7}$$

The discretization of the saddle-point of the problem (3.1) by introduce a partition $(t_0, t_1, \ldots, t_N)$ of time interval $[0, T]$ and consider on incremental formulation obtained by using the backward finite difference approximation of the time derivative of u.

If we used $u_h^k = u_h(x, t_k)$, $\Delta u_h^k = u_h^{k+1} - u_h^k$, $\Delta t^k = t^{k+1} - t^k$, $\dot{u}_h(t^{k+1}) = \Delta u_h^k/\Delta t$, $f_h^k = f_h(k\Delta t)$, $h_h^k = h_h(k\Delta t)$, $\sigma_h^k = \sigma_h(u_h^k)$, $\lambda_h^k = |u_{Th}^k|$, for $k = 0, 1 \ldots, N$

where $\Delta t = \dfrac{T}{N}$.
In this case, we find $(\boldsymbol{\sigma}_h^k, \boldsymbol{u}_h^k, \boldsymbol{\lambda}_h^k) \in \mathcal{S}_t^h \times V_h \times \Lambda_h$ such that

$$L(\boldsymbol{\sigma}_h^k, \boldsymbol{v}_h^k, \boldsymbol{\mu}_h^k) \leq L(\boldsymbol{\sigma}_h^k, u_h^k, \lambda_h^k) \leq L(\boldsymbol{\tau}_h^k, u_h^k, \lambda_h^k),\ \forall\ (\boldsymbol{\tau}_h^k, \boldsymbol{v}_h^k, \boldsymbol{\mu}_h^k) \in S_0^h \times V_h \times \Lambda_h, \quad (3.8)$$

$k = 0, 1 \ldots, N$.

In this mode the quasi-static problem is approximated by a sequence of incremental problems (3.8).

Although, every problem (3.2) is a static one, it requires appropriate updating of the displacements and the loads after each increment.
The existence of the solution is guaranteed by the discrete Babuška-Brezzi condition should by satisfied for dual mixed element method, see ([4] and [14]).

4. **Convergence analysis of the Uzawa algorithm**

On the convergence (see [11]) with the finite element discrete problem (3.1) is following:

Proposition 4.1. *If* $(\boldsymbol{\sigma}_h^k, \boldsymbol{u}_h^k, \boldsymbol{\lambda}_h^k)$ *is the saddle-point of the problem* (3.8)*, then*

$$(i)\qquad J_0(\boldsymbol{\sigma}_h^k, \boldsymbol{u}_h^k) - (\mathrm{div}\,\boldsymbol{\sigma}_h^k, \boldsymbol{u}_h^k) - < \mu_F \left|\boldsymbol{\sigma}_N^k\right|,\ \boldsymbol{\lambda}_h^k >_{\Gamma_c} \leq$$
$$\leq J_0(\boldsymbol{\tau}_h^k, \boldsymbol{u}_h^k) - (\mathrm{div}\,\boldsymbol{\tau}_h^k, \boldsymbol{u}_h^k) - < \mu_F \left|\boldsymbol{\tau}_N^k\right|,\ \boldsymbol{\lambda}_h^k >_{\Gamma_c},\ (\forall\, \boldsymbol{\tau}_h^k \in S_0^h),$$
$$(ii)\qquad < \mu_F \left|\boldsymbol{\sigma}_N^k\right|,\ \boldsymbol{\mu}_h^k - \boldsymbol{\lambda}_h^k >_{\Gamma_c} + (\mathrm{div}\,\boldsymbol{\sigma}_h^k + \boldsymbol{f}^k,\ \boldsymbol{v}_h^k - \boldsymbol{u}_h^k) \leq 0,$$
$$(\forall\, \boldsymbol{\mu}_h^k \in \Lambda_h,\ \boldsymbol{v}_h^k \in V_h)$$

where $\boldsymbol{\lambda}_h^k = |\boldsymbol{v}_{Th}^k|, \boldsymbol{\mu}_h = |\boldsymbol{u}_{Th}^k|$ *on* Γ_c*,* $k = 0, 1, \ldots, N$*.*

The proof can be deduced directly from the two inequalities showed at (3.8).

Proposition 4.2. *The variational problem*

$$(\mathrm{div}\,\boldsymbol{\sigma}_h^k + \boldsymbol{f}^k, \boldsymbol{v}_h^k - \boldsymbol{u}_h^k) + < \mu_F \left|\boldsymbol{\sigma}_N^k\right|, \boldsymbol{\mu}_h^k - \boldsymbol{\lambda}_h^k >_{\Gamma_c} \leq 0\ (\forall\, \boldsymbol{\mu}_h^k \in \Lambda_h, \boldsymbol{v}_h^k \in V_h) \qquad (4.1)$$

is equivalent to

$$\mathrm{div}\boldsymbol{\sigma}_h^k + \boldsymbol{f}_h^k = 0, \quad \boldsymbol{\lambda}_h^k = P_\Lambda(\rho \boldsymbol{s}_h^k + \boldsymbol{\lambda}_h^k) \qquad (4.2)$$

where P_Λ *is the projection operator from* $L^2(\Gamma_c)$ *to* Λ_h *is the convex subset of* $H^{1/2}(\Gamma_c), \rho > 0$*,* $\boldsymbol{s}_h^k = \mu_F \left|\boldsymbol{\sigma}_N^k\right|, k = 0, 1, \ldots, N$*.*

Proof. The inequation (4.1) is equivalent to

$$(\mathrm{div}\boldsymbol{\sigma}_h^k + \boldsymbol{f}^k, \boldsymbol{u}_h^k - \boldsymbol{v}_h^k) + \boldsymbol{s}_h^k, \boldsymbol{\lambda}_h^k - \boldsymbol{\mu}_h^k >_{\Gamma_c} \geq 0\ (\forall\, \boldsymbol{\mu}_h^k \in \Lambda_h, \boldsymbol{v}_h^k \in V_h). \qquad (4.3)$$

Multiplying the inequation (4.3) by ρ and adding $(\boldsymbol{u}_h^k - \boldsymbol{v}_h^k, \boldsymbol{u}_h^k)$ to the two sides of (4.3), we have

$$(\boldsymbol{u}_h^k - \boldsymbol{v}_h^k, \rho(\mathrm{div}\,\boldsymbol{\sigma}_h^k + \boldsymbol{f}^k) + \boldsymbol{u}_h^k) + < \boldsymbol{\lambda}_h^k - \boldsymbol{\mu}_h^k, \rho \boldsymbol{s}_h^k + \boldsymbol{\lambda}_h^k >_{\Gamma_c} \geq$$
$$\geq (\boldsymbol{u}_h^k - \boldsymbol{v}_h^k, \boldsymbol{u}_h^k) + < \boldsymbol{\lambda}_h^k - \boldsymbol{\mu}_h^k, \boldsymbol{\lambda}_h^k >_{\Gamma_c}. \qquad (4.4)$$

But P_Λ is a projector operator,

$$(\boldsymbol{u}_h^k - \boldsymbol{v}_h^k, \rho(\mathrm{div}\,\boldsymbol{\sigma}_h^k + \boldsymbol{f}_h^k) + \boldsymbol{u}_h^k) + (\boldsymbol{\lambda}_h^k - \boldsymbol{\mu}_h^k, P_\Lambda(\rho \boldsymbol{s}_h^k + \boldsymbol{\lambda}_{\boldsymbol{h}}^{\boldsymbol{k}}))_{0,\Gamma_c} \geq$$
$$\geq (\boldsymbol{u}_h^k - \boldsymbol{v}_h^k, \boldsymbol{u}_h^k) + < \boldsymbol{\lambda}_h^k - \boldsymbol{\mu}_h^k, \boldsymbol{\lambda}_h^k >_{\Gamma_c}.$$

Hence

$$(\boldsymbol{u}_h^k-\boldsymbol{v}_h^k,\rho(\operatorname{div}\boldsymbol{\sigma}_h^k+\boldsymbol{f}_h^k))+(\boldsymbol{\lambda}_h^k-\boldsymbol{\mu}_h^k,P_\Lambda(\rho\boldsymbol{s}_h^k+\boldsymbol{\lambda}_h^k)-\boldsymbol{\lambda}_h^k)_{0,\Gamma_c}\geq 0\ . \tag{4.5}$$

Because V_h and Λ_h are convex sets, we can put $(0<\alpha<1)$:

$$\left.\begin{array}{rcl}\boldsymbol{v}_h^k &=& (1-\alpha)\boldsymbol{u}_h^k+\alpha(\rho(\operatorname{div}\boldsymbol{\sigma}_h^k+\boldsymbol{f}_h^k)+\boldsymbol{u}_h^k)\\ \boldsymbol{\mu}_h^k &=& (1-\alpha)\boldsymbol{\lambda}_h^k+\alpha P_\Lambda(\rho\boldsymbol{s}_h^k+\boldsymbol{\lambda}_h^k)\end{array}\right\}. \tag{4.6}$$

Substituting (4.6) in (4.5) yields

$$\alpha(-\rho(\operatorname{div}\boldsymbol{\sigma}_h^k+\boldsymbol{f}_h^k),\rho(\operatorname{div}\boldsymbol{\sigma}_h^k+\boldsymbol{f}_h^k))+\alpha(\boldsymbol{\lambda}_h^k-P_\Lambda(\rho\boldsymbol{s}_h^k+\boldsymbol{\lambda}_h^k),P_\Lambda(\rho\boldsymbol{s}_h^k+\boldsymbol{\lambda}_h^k)-\boldsymbol{\lambda}_h^k)_{0,\Gamma_c}\geq 0\ ,$$

that is equivalent with

$$\alpha\|\rho(\operatorname{div}\boldsymbol{\sigma}_h^k+\boldsymbol{f}_h^k)\|_{0,\Omega}^2+\alpha\|\boldsymbol{\lambda}_h^k-P_\Lambda(\rho\boldsymbol{s}_h^k+\boldsymbol{\lambda}_h^k)\|_{0,\Gamma_c}^2\leq 0\ (0<\alpha<1,\ \rho>0)\ ,$$

so we obtain

$$\operatorname{div}\boldsymbol{\sigma}_h^k+\boldsymbol{f}_h^k=0 \quad\text{and}\quad \boldsymbol{\lambda}_h^k=P_\Lambda(\rho\boldsymbol{s}_h^k+\boldsymbol{\lambda}_h^k),\ \rho>0,\ k=0,1,\ldots,N.$$

From this results we can define the following Uzawa algorithm type:

a) Given $\boldsymbol{u}_h^{nk}\in V_h,\boldsymbol{\lambda}_h^{nk}\in\Lambda_h$, we can define $\boldsymbol{\sigma}_h^{nk}\in S_t^h$ such that

$$\begin{aligned}&J_0(\boldsymbol{\sigma}_h^{nk})-(\operatorname{div}\boldsymbol{\sigma}_h^{nk},\boldsymbol{u}_h^{nk})-<\boldsymbol{s}_h^{nk},\boldsymbol{\lambda}_h^{nk}>_{\Gamma_c}\leq\\ &\qquad\leq J_0(\boldsymbol{\tau}_h^{nk})-(\operatorname{div}\boldsymbol{\tau}_h^{nk},\boldsymbol{u}_h^{nk})+<\boldsymbol{t}_h^{nk},\boldsymbol{\lambda}_h^{nk}>_{\Gamma_c},\ \forall\,\boldsymbol{\tau}_h^{nk}\in S_0^h;\end{aligned} \tag{4.7}$$

b) Find $\boldsymbol{u}_h^{(n+1)k}$ and $\boldsymbol{\lambda}_h^{(n+1)k}=\left|\boldsymbol{v}_{Th}^{(n+1)k}\right|$ by using the following iterative method:

$$\boldsymbol{u}_h^{(n+1)k}=\boldsymbol{u}_h^{nk}+\rho_n(\operatorname{div}\boldsymbol{\sigma}_h^{nk}+\boldsymbol{f}^k) \tag{4.8}$$

$$\boldsymbol{\lambda}_n^{(n+1)k}=P_\Lambda(\boldsymbol{\rho}_n\boldsymbol{s}_h^{nk}+\boldsymbol{\lambda}_h^{nk}), \tag{4.9}$$

when $\rho_n>0$ is chosen properly, $k=0,1,\ldots,N$. □

We define the following bounded linear operator: $g_\tau : S_h\to V\times L^2(\Gamma_c)$ by

$$g_\tau(\boldsymbol{v},\boldsymbol{\mu})=(\operatorname{div}\boldsymbol{\tau},\boldsymbol{\nu})+<\boldsymbol{s},\boldsymbol{\mu}>_{\Gamma_c},\quad s=\mu_F|\boldsymbol{\sigma}_N(\boldsymbol{v})|,\ \boldsymbol{\mu}=|\boldsymbol{v}_T|\ .$$

Proposition 4.3. *The operator $g_\tau : S_h\to V\times L^2(\Gamma_c)$ is Lipschitz continuous, i.e. there exists a constant $c>0$, such that*

$$\|g_\tau(\boldsymbol{\tau}_1)-g_\tau(\boldsymbol{\tau}_2)\|_{V\times L^2(\Gamma_c}\leq c\|\boldsymbol{\tau}_1-\boldsymbol{\tau}_2\|_s,\quad\forall\,\boldsymbol{\tau}_1,\boldsymbol{\tau}_2\in S_h,$$

where $\|\cdot\|_{V\times L^2(\Gamma_c)}$ denotes the norm of product space $V\times L^2(\Gamma_c)$.

Proof is obtained from definition of g_r and from (3.7).

Theorem 4.4. *There exists the constant α_0 and α_1, with $0<\alpha_0\leq\rho_n\leq\alpha_1$, such that, the Uzawa type algorithm a)-b), is convergent in sense that $\boldsymbol{\sigma}_h^{nk}\to\boldsymbol{\sigma}_h^k$ strongly in S.*

Proof. We denote $\boldsymbol{r}_1^{nk}=\boldsymbol{u}_h^{nk}-\boldsymbol{u}_h^k$, $\boldsymbol{r}_2^{nk}=\boldsymbol{\lambda}_h^{nk}-\boldsymbol{\lambda}_h^k$, and from (4.7)-(4.9) we can deduce:

$$\begin{aligned}
\|\boldsymbol{r}_1^{(n+1)k}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{(n+1)k}\|_{0,\Gamma_c}^2&=\|\boldsymbol{u}_h^{(n+1)k}-\boldsymbol{u}_h^k\|_{0,\Omega}^2+\|\boldsymbol{\lambda}_h^{(n+1)k}-\boldsymbol{\lambda}_h^k\|_{0,\Gamma_c}^2=\\
&=\big\|\boldsymbol{u}_h^{nk}+\rho_n(\mathrm{div}\,\boldsymbol{\sigma}_h^{nk}+\boldsymbol{f}^k)-\boldsymbol{u}_h^{nk}-\rho_n(\mathrm{div}\,\boldsymbol{\sigma}_h^{nk}+\boldsymbol{f}^k)\big\|_{0,\Omega}^2+\\
&+\big\|P_\Lambda(\rho_n\boldsymbol{s}_h^{nk}+\boldsymbol{\lambda}_h^{nk})-P_\Lambda(\rho_n\boldsymbol{s}_h^{nk}+\boldsymbol{\lambda}_h^{nk}\big\|_{0,\Gamma_c}^2\le\\
&\le\big\|\boldsymbol{r}_1^{nk}+\rho_n\mathrm{div}\,(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k)\big\|_{0,\Omega}^2+\big\|\rho_n(\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k)+(\boldsymbol{\lambda}_h^{nk}-\boldsymbol{\lambda}_h^k)\big\|_{0,\Gamma_c}^2=\\
&=\|\boldsymbol{r}_1^{nk}\|_{0,\Omega}^2+2\rho_n(\boldsymbol{r}_1^{nk},\mathrm{div}(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k))+\rho_n^2\|\mathrm{div}(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k)\|_{0,\Omega}^2+\\
&+\|\boldsymbol{r}_2^{nk}\|_{0,\Gamma_c}^2+2\rho_n(\boldsymbol{r}_2^{nk},\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k)_{0,\Gamma_c}+\boldsymbol{\rho}_n^2\|\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k\|_{0,\Omega}^2=\\
&=\|\boldsymbol{r}_1^{nk}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{nk}\|_{0,\Gamma_c}^2+2\rho_n(\boldsymbol{r}_1^{nk},\ \mathrm{div}(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k))+(\boldsymbol{r}_2^{nk},\ (\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k))_{0,\Gamma_c}+\\
&+\rho_n^2\|\mathrm{div}(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k)\|_{0,\Omega}^2+\|\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k\|_{0,\Gamma_c}^2\,. \qquad (4.10)
\end{aligned}$$

With the Proposition 4.3 and (4.10) can be regarded as positive algebraic equations with degree two in ρ, we get

$$a(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k,\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k)+(\boldsymbol{r}_1^{nk},\mathrm{div}(\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k))+<\boldsymbol{r}_2^{nk},\boldsymbol{s}_h^{nk}-\boldsymbol{s}_h^k>_{\Gamma_c}\le 0\,,$$

where a is a linear symmetric form $a:S\times S\to\mathbb{R}$, which with (4.10) implying:

$$\begin{aligned}
\|\mathbf{r}_1^{(n+1)k}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{(n+1)k}\|_{0,\Gamma_c}^2&\le\|\boldsymbol{r}_1^{nk}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{nk}\|_{0,\Gamma_c}^2-\\
-2\rho_n a(\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h,\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_{\boldsymbol{h}}^{\boldsymbol{k}})&+2\rho_n^2\|\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k\|_S^2\le\\
\le\|\boldsymbol{r}_1^{nk}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{nk}\|_{0,\Gamma_c}^2&-(2\rho_n-\rho_n^2)\|\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k\|_S^2\,.
\end{aligned}$$

For this inequation, we suppose $2\rho_n-2\rho_n^2\ge\beta>0$, and we choose $\alpha_0=\dfrac{1-\sqrt{1-2\beta}}{2}$, $\alpha_1=\dfrac{1+\sqrt{1-2\beta}}{2}$ such that for $\rho_n\in[\alpha_0,\alpha_1]$, then we have:

$$\|\boldsymbol{r}_1^{(n+1)k}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{(n+1)k}\|_{0,\Gamma_c}^2+\beta\|\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k\|_S^2\le\|\boldsymbol{r}_1^{nk}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{nk}\|_{0,\Gamma_c}^2 \qquad (4.11)$$

From (4.11) results that the sequence $\left(\|\boldsymbol{r}_1^{nk}\|_{0,\Omega}^2+\|\boldsymbol{r}_2^{nk}\|_{0,\Gamma_c}^2\right)_n$ is decreasing and has a finite limit, so that $\beta\|\boldsymbol{\sigma}_h^{nk}-\boldsymbol{\sigma}_h^k\|_S^2\to 0$ for $n\to\infty$, and Theorem 4.4 is proved. ☐

The solution $\boldsymbol{\sigma}_h^k$ of (3.8) is a fixed point of function $M_h:S_h\to S_h$, so that $\boldsymbol{\sigma}_h^k$ is the limit of a sequence $(\boldsymbol{\sigma}_h^{nk})_n$, defined by $\boldsymbol{\sigma}_h^{nk}=M_h\boldsymbol{\sigma}_h^{(n-1)k}$, (see [13]).

Theorem 4.5. *In the conditions of Theorem* 4.4, *if* $\alpha_0<\rho_n<\alpha_1$ *is true* (α_1 *are chosen according to Theorem* 4.4, *then for the sequences* $\{\boldsymbol{u}_h^{nk}\}_n$, $\{\boldsymbol{\lambda}_h^{nk}\}_n$ *defined by* (4.8) – (4.9) *we have:*

a) $\lim\limits_{n\to\infty}\|\boldsymbol{u}_h^{(n+1)k}-\boldsymbol{u}_h^k\|_{0,\Omega}=0$, $\lim\limits_{n\to\infty}\|\boldsymbol{\lambda}_h^{(n+1)k}-\boldsymbol{\lambda}_h^k\|_{0,\Gamma_c}=0$;

b) $\{\boldsymbol{u}_h^{nk},\boldsymbol{\lambda}_h^{nk}\}_n\to\{u_h,\lambda_h\}$ *weakly in* $V_h\times\Lambda_h$ *where* $\{\boldsymbol{u}_h^k,\boldsymbol{\lambda}_h^k\}$ *is such that* $\boldsymbol{\sigma}_h^k,\boldsymbol{u}_h^k,\boldsymbol{\lambda}_h^k$ *is a saddle-point of* $L(\boldsymbol{\tau}_h^k,\boldsymbol{v}_h^k,\boldsymbol{\mu}_h^k)$ *on* $S_t^h\times V_h\times\Lambda_h$.

The proof is similar to that of Theorem 4.4, see [3].

5. **Conclusions**

We have analyzed, with Uzawa type algorithm of dual mixed variational formulation of the reduced version of a contact problem with friction in which it is assumed that the normal contact component of stress vector is known. For a more general contact problem, the existence solution is proved, but in very special cases.

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