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CARISTI TYPE OPERATORS AND APPLICATIONS

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Dedicated to Professor Gheorghe Micula at his 60^{th} *anniversary*

1. Introduction

Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X \tag{1.1}$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$. (see Caristi [4]).

For the multi-valued case, there exist several results involving multi-valued Caristi type conditions. For example, if F is a multi-valued operator from a complete metric space (X, d) into itself and if there exists a proper, lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x, y) + \varphi(y) \leq \varphi(x), \tag{1.2}$$

then the multi-valued map F has at least a fixed point $x^* \in X$, i. e. $x^* \in F(x^*)$. (see Mizoguchi-Takahashi [11]).

Moreover, if F satisfies the stronger condition:

$$\text{for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x, y) + \varphi(y) \leq \varphi(x), \tag{1.3}$$

then the multi-valued map F has at least a strict fixed point $x^* \in X$, i. e. $\{x^*\} = F(x^*)$. (see Maschler-Peleg [10]).

Another result of this type was proved by L. van Hot, as follows.

If F is a multi-valued operator with nonempty closed values and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower semi-continuous function such that the following condition holds:

$$\text{for each } x \in X, \text{ inf } \{ \ d(x, y) + \varphi(y) \ : \ y \in F(x) \ \} \leq \varphi(x), \tag{1.4}$$

then F has at least a fixed point. (see van Hot [6])

There are several extensions and generalizations of these important principles of nonlinear analysis (see the references list and also the bibliography therein)

The purpose of this paper is to present several new results and open problems for single-valued and multi-valued Caristi type operators between metric spaces. Also,

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the case of metric spaces endowed with a weakly distance in the sense of Kada-Suzuki-Takahashi is considered.

2. Preliminaries

Throughout this paper (X, d) is a complete metric space, $f : X \to X$ is a single-valued operator and $F : X \multimap X$ denotes a multi-valued operator.

Definition 2.1. *A single-valued operator $f : X \to X$ is said to be:*

a) Caristi type operator if there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in .$$

b) Kannan type operator if there exists $a \in [0, \frac{1}{2}[$ such that

$$d(f(x), f(y)) \leq a[d(x, f(x)) + d(y, f(y))], \text{ for each } x, y \in X.$$

c) Ciric-Reich-Rus type operator if there exist $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$ such that

$$d(f(x), f(y)) \leq ad(x, y) + bd(x, f(x)) + cd(y, f(y)), \text{ for each } x, y \in X.$$

If $b = c = 0$, then f is called an a-contraction.

Definition 2.2. *If $f : X \to X$ is a single-valued operator, then $x^* \in X$ is called a fixed point of f if $x^* = f(x^*)$. We will denote by $Fixf$ the fixed point set of f.*

If (X, d) is a metric space, then $\mathcal{P}(X)$ will denote the space of all subsets of X. Also, we denote by $P(X)$ the space of all nonempty subsets of X and by $P_p(X)$ the set of all nonempty subsets of X having the property "p", where "p" could be: *cl = closed*, *b = bounded*, *cp = compact*, *cv = convex* (for normed spaces X), etc.

We consider the following (generalized) functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+, \ D(A, B) = \inf \{ \ d(a, b) \mid a \in A, \ b \in B \ \}$$

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \ H(A, B) = \max \{ \sup_{a \in A} D(a, B), \ \sup_{b \in B} D(b, A) \ \}.$$

H is called the Pompeiu-Hausdorff generalized functional and it is well-known that if (X, d) is a complete metric space, then $(P_{cl}(X), H)$ is also a complete metric space.

Definition 2.3. *Let (X, d) be a metric space. Then a multi-valued operator $F : X \to P(X)$ is called:*

a) (M-T)- Caristi type multifunction if there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x, y) + \varphi(y) \leq \varphi(x)$$

(see Mizoguchi-Takahashi [11]*)*

b) (M-P)- Caristi type multifunction if there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x, y) + \varphi(y) \leq \varphi(x).$$

(see Maschler-Peleg [10]*)*

c) (vH)- Caristi type multifunction if F has closed values and there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X,\ \inf\ \{\ d(x,y) + \varphi(y)\ :\ y \in F(x)\ \} \leq \varphi(x)$$

(see van Hot [6])

d) Kannan type multifunction if there exists $a \in [0, \frac{1}{2}[$ such that

$$H(F(x), F(y)) \leq a[D(x, F(x)) + D(y, F(y))],\ \text{for each } x, y \in X.$$

e) Reich type multifunction if there exist $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$ such that

$$H(F(x), F(y)) \leq ad(x,y) + bD(x, F(x)) + cD(y, F(y)),\ \text{for each } x, y \in X.$$

If $b = c = 0$, then F is called a multi-valued a-contraction.

Remark 2.1. *It is quite obviously that if F satisfies a (M-P)- Caristi type condition then F is a (M-T)- Caristi type multifunction and any (M-T)- Caristi type multifunction satisfies a (vH)- Caristi type condition.*

Definition 2.4. *Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued map. Then an element $x^* \in X$ is called a fixed point of F if $x^* \in F(x^*)$ and it is called a strict fixed point of F if $\{x^*\} = F(x^*)$. We denote by $Fix(F)$ the fixed points set of F and by $SFix(F)$ the strict fixed points set of F.*

In 1996, Kada, Suzuki and Takahashi introduced the concept of w-distance on a metric space as follows.

Definition 2.5. *Let (X, d) be a metric space. Then a function $p : X \times X \to \mathbb{R}_+$ is called a w-distance on X if the following are satisfied:*

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$

(2) for any $x \in X$, $p(x, \cdot) : X \to \mathbb{R}_+$ is lower semicontinuous

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Some examples of w-distances are:

Example 2.1. *Let X be a metric space with metric d. Then $p = d$ is a w-distance on X.*

Example 2.2. *Let X be a normed space with norm $\|\cdot\|$. Then the function $p : X \times X \to \mathbb{R}_+$ defined by $p(x, y) = \|x\| + \|y\|$, for every $x, y \in X$ is a w-distance on X.*

Example 2.3. *Let X be a metric space with metric d and let T be a continuous mapping from X into itself. Then a function $p : X \times X \to \mathbb{R}_+$ defined by*

$$p(x, y) = max(d(Tx, y), d(Tx, Ty)),\ \text{for every } x, y \in X$$

is a w-distance on X.

Example 2.4. *Let $X = \mathbb{R}$ with the usual metric and $f : \mathbb{R} \to \mathbb{R}_+$ be a continuous function such that*

$$\inf_{x \in X} \int_x^{x+r} f(u)du > 0,\ \text{for each } r > 0.$$

Then a function $p : X \times X \to \mathbb{R}_+$, *defined by*

$$p(x,y) := |\int_x^y f(u)du|, \text{ for every } x, y \in X$$

is a w-distance on X.

For other examples and related results, see Kada, Suzuki and Takahashi [7].

Some important properties of the w-distance are contained in:

Lemma 2.1. *Let* (X, d) *be a metric space and* p *be a w-distance on* X. *Let* $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ *be sequences in* X, *let* $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ *be sequences in* $\mathbb{R}_+$ *converging to* 0 *and let* $y, z \in X$. *Then the following hold:*

(i) if $p(x_n, x_m) \leq \alpha_n$, *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* (x_n) *is a Cauchy sequence.*

(ii) if $p(y, x_n) \leq \alpha_n$, *for any* $n \in \mathbb{N}$, *then* (x_n) *is a Cauchy sequence.*

(iii) if $p(x_n, y_n) \leq \alpha_n$ *and* $p(x_n, z) \leq \beta_n$, *for any* $n \in \mathbb{N}$ *then* (y_n) *converges to* z.

3. **Single-valued Caristi type operators**

If $f : X \to X$ is an a-contraction, then it is well-known (see for example Dugundji-Granas [5]) that f is a Caristi type operator with a function $\varphi(x) = \frac{1}{1-a}d(x, f(x))$. Also, Caristi type mappings include Reich type operators and in particular Kannan operators. Indeed, if f satisfies a Reich type condition with constants a, b, c, then f is a Caristi type operator with a function $\varphi(x) = \frac{1-c}{1-a-b-c}d(x, f(x))$.

Moreover, if $f : X \to X$ satisfies the following condition (see I. A. Rus (1972), [14]):

there is $a \in [0, 1[$ such that $d(f(x), f^2(x)) \leq ad(x, f(x))$, for each $x \in X$

then f is a Caristi operator with a function $\varphi(x) = \frac{1}{1-a}d(x, f(x))$.

Hence, the class of single-valued Caristi type operators is very large, including at least the above mentioned types of contractive mappings.

Some characterizations of metric completeness have been discussed by several authors such as Weston, Kirk, Suzuki, Suzuki and Takahashi, Shioji, Suzuki and Takahashi, etc. For example, Kirk [8] proved that a metric space is complete if it has the fixed point property for Caristi mappings. Moreover, Shioji, Suzuki and Takahashi proved in [15] that a metric space is complete if and only if it has the fixed point property for Kannan mappings. On the other hand, it is well-known that the fixed point property for a-contraction mappings does not characterize metric completeness, see for example Suzuki-Takahashi [16]. Thus, Kannan mappings and Caristi mappings characterize metric completeness, while contraction mappings do not. Regarding to the problem of characterizations of metric completeness by means of contraction mappings, Suzuki and Takahashi and independently M. C. Anisiu and V. Anisiu showed (see [16] respectively [1]) that a convex subset Y of a normed space is complete if and only if every contraction $f : Y \to Y$ has a fixed point in Y.

The following generalization of Caristi's theorem is proved in Kada-Suzuki-Takahashi [7]:

Theorem 3.1. *Let (X, d) be a complete metric space, let $\varphi : X \times X \to \mathbb{R}_+$ be a proper lower semicontinuous function and let $f : X \to X$ a mapping. Assume that there exists a w-distance p on X such that*

$$p(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X.$$

Then there exists $x^* \in X$ such that $x^* = f(x^*)$ and $p(x^*, x^*) = 0$.

Open problem. The following result (see Zhong, Zhu and Zhao [18]) is a generalization of Caristi's fixed point principle:

Theorem 3.2. *If (X, d) is a complete metric space, $x_0 \in X$, $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ is proper lower semicontinuous and $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\int_0^\infty \frac{ds}{1+h(s)} = \infty$, then each single-valued operator f from X to itself satisfying the condition:*

$$\text{for each } x \in X, \frac{d(x, f(x))}{1 + h(d(x_0, x))} + \varphi(f(x)) \leq \varphi(x), \tag{3.5}$$

has at least a fixed point.

It is of interest to see if such a result, in terms of w-distances, is true.

4. Multi-valued Caristi type operators

It was proved by L. van Hot that any multi-valued a-contraction F on a metric space X is a (vH)-Caristi type multi-function with a function $\varphi(x) = \frac{1}{1-a} D(x, F(x))$. Moreover, if is a multi-valued a-contraction with nonempty and compact values then F satisfies a (M-T)- Caristi type condition with a same function $\varphi(x) = \frac{1}{1-a} D(x, F(x))$.

Let us remark now, that any Reich type multi-function (and hence in particular any Kannan multi-function) is a (vH)- Caristi type multi-function with a function φ given by $\varphi(x) = \frac{1-c}{1-a-b-c} D(x, F(x))$.

Definition 4.6. *If (X, d) is a metric space, then a multi-valued operator $F : X \to P(X)$ is said to be a Reich type graphic contraction if there exist $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$ such that*

$$H(F(x), F(y)) \leq ad(x, y) + bD(x, F(x)) + cD(y, F(y)),$$

for each $x \in X$ and each $y \in F(x)$.

A connection between multi-valued Reich type graphic contractions and multi-valued Caristi type operators is given in:

Lemma 4.2. *Let (X, d) be a metric space and let $F : X \to P(X)$ be a Reich type graphic contraction. Then F is a (vH)- Caristi type multi-function.*

Proof. Let $\varphi(x) = \frac{1-c}{1-a-b-c} D(x, f(x))$. Then, because for each $x \in X$ and each $y \in F(x)$ we have $D(y, F(y)) \leq H(F(x), F(y)) \leq ad(x, y) + bD(x, F(x)) + cD(y, F(y))$, we obtain $D(y, F(y)) \leq \frac{1}{1-c}(ad(x, y) + bD(x, F(x)))$.

Hence for $x \in X$ and $y \in F(x)$ we get $d(x, y) + \varphi(y) \leq d(x, y) + \frac{1}{1-a-b-c}(ad(x, y) + bD(x, F(x)))$ and so $inf\{d(x, y) + \varphi(y) | y \in F(x)\} \leq inf\{\frac{1-b-c}{1-a-b-c} d(x, y) | y \in F(x)\} + \frac{b}{1-a-b-c} D(x, F(x)) = \varphi(x)$. In conclusion, F is a (vH)- Caristi type multi-function and the proof is complete. ■

For the case of complete metric spaces endowed with a w-distance the following generalization of the Covitz-Nadler fixed point principle for multi-functions was proved by Suzuki and Takahashi in [16]. We need, first, a definition.

Definition 4.7. *Let (X, d) be a metric space. A multi-valued mapping $F : X \to P(X)$ is called p-contractive if there exist a w-distance p on X and a real number $a \in [0, 1[$ such that for any $x_1, x_2 \in X$ and each $y_1 \in F(x_1)$ there exists $y_2 \in F(x_2)$ so that $p(y_1, y_2) \leq ap(x_1, x_2)$*

Theorem 4.3. *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $F : X \to P_{cl}(X)$ be a p-contractive multi-function. Then there exists $x^* \in X$ a fixed point for F and $p(x^*, x^*) = 0$.*

Some extensions of the previous result are:

Theorem 4.4. *Let (X, d) be a complete metric space and $F : X \to P_{cl}(X)$ be a closed multi-valued operator such that the following assumption holds:*

there exist a w-distance p on X and a real number $a \in [0, 1[$ so that for any $x \in X$ and any $y_1 \in F(x)$ there is $y_2 \in F(y_1)$ such that $p(y_1, y_2) \leq ap(x, y_1)$.

Then there exists $x^* \in X$ a fixed point for F and $p(x^*, x^*) = 0$.

Proof. Let $u_0 \in X$ and $u_1 \in F(u_0)$. Then there exists $u_2 \in F(u_1)$ such that $p(u_1, u_2) \leq ap(u_0, u_1)$. Then, $u_1 \in X$ and $u_2 \in F(u_1)$ we obtain that there is $u_3 \in F(u_2)$ with $p(u_2, u_3) \leq ap(u_1, u_2) \leq a^2 p(u_0, u_1)$. Thus we can construct a sequence $(u_n)_{n\in\mathbb{N}}$ from X satisfying:

i) $u_{n+1} \in F(u_n)$, for each $n \in \mathbb{N}$.

ii) $p(u_n, u_{n+1}) \leq a^n p(u_0, u_1)$, for each $n \in \mathbb{N}$.

Hence, for any $n, m \in \mathbb{N}$ with $m > n$ we have: $p(u_n, u_m) \leq p(u_n, u_{n+1}) + ... + p(u_{m-1}, u_m) \leq a^n p(u_0, u_1) + a^{n+1} p(u_0, u_1) + ... + a^{m-1} p(u_0, u_1) \leq \frac{a^n}{1-a} p(u_0, u_1)$. By Lemma 2.1 $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence $(u_n)_{n\in\mathbb{N}}$ converges to a point $u^* \in X$. Since F is closed we get that $u^* \in F(u^*)$.

Let us consider now a fixed $n \in \mathbb{N}$. Since $(u_m)_{m\in\mathbb{N}}$ converges to a $u^* \in X$ and $p(u_n, \cdot)$ is lower semicontinuous we have

$$p(u_n, u^*) \leq \liminf_{m\to\infty} p(u_n, u_m) \leq \frac{a^n}{1-a} p(u_0, u_1).$$

For $u^* \in F(u^*)$, there exists $w_1 \in F(u^*)$ such that $p(u^*, w_1) \leq ap(u^*, u^*)$. By a similar approach, we can construct a sequence $(w_n)_{n\in\mathbb{N}}$ in X such that $w_{n+1} \in F(w_n)$ and $p(u^*, w_{n+1}) \leq ap(u^*, w_n)$, for each $n \in \mathbb{N}$. So, as before we obtain

$$p(u^*, w_n) \leq ap(u^*, w_{n-1}) \leq ... \leq a^n p(u^*, u^*).$$

By Lemma 2.1, (w_n) is a Cauchy sequence and it converges to a point $x^* \in X$. Since $p(u^*, \cdot)$ is lower semicontinuous we have

$$p(u^*, x^*) \leq \liminf_{n\to\infty} p(u^*, w_n) \leq 0$$

and hence $p(u^*, x^*) = 0$. Then, for any $n \in \mathbb{N}$ we have

$$p(u_n, x^*) \leq p(u_n, u^*) + p(u^*, x^*) \leq \frac{a^n}{1-a} p(u_0, u_1).$$

Using again Lemma 2.1 we obtain $u^* = x^*$ and hence $p(x^*, x^*) = 0$. The proof is now complete. ■

Theorem 4.5. *Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $F : X \to P_{cl}(X)$ be a closed multi-valued operator having the following property:*

there exists a w-distance p on X so that for each $x \in X$ there is $y \in F(x)$ we have $p(x, y) + \varphi(y) \leq \varphi(x)$.

Then $FixF \neq \emptyset$.

Moreover, if F satisfies the stronger condition:

there exists a w-distance p on X so that for each $x \in X$ and for each $y \in F(x)$ we have $p(x, y) + \varphi(y) \leq \varphi(x)$,

then there exists $x^* \in X$ a fixed point for T and $p(x^*, x^*) = 0$.

Proof. From the hypothesis it follows that if p is a w-distance on (X, d), there exists a single-valued operator $f : X \to X$ such that f is a selection for F and satisfies the condition

$$p(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X.$$

The first conclusion follows now from Theorem 3.1. For the second conclusion of the theorem, we observe that for $x^* \in F(x^*)$ we have $p(x^*, x^*) + \varphi(x^*) \leq \varphi(x^*)$ and hence $p(x^*, x^*) = 0$. The proof is now complete. ■

Following an idea from Kirk, Srivassan and Veeramani [9] we also have the following Caristi type theorem:

Theorem 4.6. *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multi-function. Let us suppose that there exist $A_1, A_2, ..., A_p$ closed subsets of X and $\varphi_i : A_i \to \mathbb{R}$, for $i \in \{1, 2, ..., p\}$ lower semicontinuous mappings, such that the following assumptions hold:*

i) $F(A_i) \subset A_{i+1}$, for $i \in \{1, 2, ..., p\}$, where $A_{p+1} = A_1$.

ii) for each $x \in A_i$ and each $y \in F(x)$ we have

$$d(x, y) + \varphi_{i+1}(y) \leq \varphi_i(x), \text{ for } i \in \{1, 2, ..., p\}.$$

Then $FixF \neq \emptyset$.

Proof. Let $x_0 \in A_1$. Then for $x_1 \in F(x_0) \subset A_2$ we have $d(x_0, x_1) \leq \varphi_1(x_0) - \varphi_2(x_1)$. Then for $x_1 \in A_2$ and $x_2 \in F(x_1) \subset A_3$ we have that $d(x_1, x_2) \leq \varphi_2(x_1) - \varphi_3(x_2)$. So, we can construct a sequence $(x_n)_{n\in\mathbb{N}}$, having the following properties:

i) $x_{n+1} \in F(x_n)$, for $n \in \mathbb{N}$.

ii) $d(x_n, x_{n+1}) \leq \varphi_{n+1}(x_n) - \varphi_{n+2}(x_{n+1})$, for $n \in \mathbb{N}$.

Let us observe that the sequence $(\varphi_{k+1}(x_k))_{k\in\mathbb{N}}$ converges to an element $a \in \mathbb{R}$. Then for $n, m \in \mathbb{N}$, with $n > m$ we get $d(x_n, x_m) \leq d(x_n, x_{n+1}) + ... + d(x_{m-1}, x_m) \leq \varphi_{n+1}(x_n) - \varphi_{m+1}(x_m)$. Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to a point $x \in \bigcap_{i=1}^{p} A_i$. So $A := \bigcap_{i=1}^{p} A_i \neq \emptyset$. Hence $F : A \to P(A)$. Moreover we can write:

for $x \in A \subset A_1$ and $y \in F(x)$ $d(x, y) \leq \varphi_1(x) - \varphi_2(y)$

for $x \in A \subset A_2$ and $y \in F(x)$ $d(x,y) \leq \varphi_2(x) - \varphi_3(y)$

...

for $x \in A \subset A_p$ and $y \in F(x)$ $d(x,y) \leq \varphi_p(x) - \varphi_1(y)$

and therefore $pd(x,y) \leq \sum_{i=1}^{p}(\varphi_i(x) - \varphi_i(y))$. If we define $\varphi : A \to \mathbb{R}$ by $\varphi(x) = \frac{1}{p}\sum_{i=1}^{p}\varphi_i(x)$, then φ is lower semicontinuous and the following assertion holds:

for each $x \in A$ there exists $y \in F(x)$ such that $d(x,y) \leq \varphi(x) - \varphi(y)$.

Using Mizoguchi-Takahashi's theorem (see [11]) we get the conclusion. ■

Remark. If in the previous theorem F is a single-valued operator then Theorem 4.6 is Theorem 3.1 from Kirk, Srivassan and Veeramani [9].

Open problem. For other Caristi type conditions, results as the previous one, involving cyclical type conditions, can be established?

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